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# ELEMENTARY MATHEMATICS.

## PART FIRST.

# ALGEBRA.

# AN ELEMENTARY COURSE

OF

# MATHEMATICS.

## FOR THE USE OF SCHOOLS.

BY JAMES ELLIOT,

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"A COMPLETE PRACTICAL TREATISE ON THE NATURE AND USE OF LOGARITHMS,
AND ON PLANE TRIGONOMETRY," ETC.

## PART FIRST.

### ALGEBRA.

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# PREFACE TO THE FOURTH EDITION OF PART FIRST.*

To those who use this work I have to apologize for again enlarging the First Part of the Course; but since, in this edition, scarcely any change has been made except that of adding to its contents, the alteration can hardly be productive of inconvenience, while it is hoped it will be welcome to those Teachers who desire to extend the range of instruction in Algebra beyond the original limits of the volume. It now includes all the branches of the subject which are usually comprehended in a University Course, and some others which utility or completeness seemed to demand. The latter, when limited time prevents indulgence in anything beyond University requirements, may be passed over, such as Chapters 27, 29, 30, and the latter part of Chapter 38. The omission of these will not, in any way, affect the study of the other chapters.

As the Fifth Book of Euclid is now usually superseded by the Algebraical method of treating ratios, I have, in Chapters 15 and 35 together, given everything that is necessary for that purpose, omitting no proposition regarding ratios contained in Euclid's Elements, advancing a little beyond his limits, and, at the same time, adding two theorems (the 16th and 17th of Chapter 35), in order to render the Algebraical method available for the demonstrations of the Sixth and subsequent Books of Euclid exactly as these stand. Chapter 35 may be taken immediately after Chapter 22, if then required for the Geometrical course which the student may be pursuing, as it depends on nothing between the two. The theorems which relate to ratios and proportion, will be found generally enunciated as nearly as possible in Euclid's words.

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* The original preface to the Course will be found in the latest edition of the Key to the Algebra, to which, as it concerns the teacher more than the pupil, it more appropriately belongs.

The subject of *Surds* has been rendered more than usually complete, and some new matter will be found under that head.

The whole volume, partly from the mode in which it has been successively enlarged, and partly from express design, has now acquired a peculiarly *progressive* character. This will doubtless be found, for purposes of instruction, to be a very great advantage. In order to obtain completeness in each branch of the subject, abstruse and less generally useful matter is often thrust in at the commencement of the course, greatly to the annoyance of the beginner; or, when that is not done, the subject is left in an imperfect state. Both evils have been avoided in this volume by introducing nothing beyond elementary matter at the commencement, and, when anything more is necessary in order to complete any particular topic, by resuming that topic again in a subsequent chapter.

J. E.

LIVERPOOL, *January* 2, 1860.

# ALGEBRA.

## CHAPTER I.

### DEFINITIONS AND EXPLANATIONS.

1. WHILE Arithmetic represents numbers by figures, Algebra represents them by letters.

2. While the figures of Arithmetic represent only *particular* numbers (each figure, or combination of figures, denoting one number only), the letters of Algebra may represent *any* numbers, or classes of numbers. Algebra is, therefore, sometimes called *Universal Arithmetic.*

3. The letters of Algebra represent not merely numbers but also quantities. The quantities, however, when subjected to Algebraical processes, are regarded as numbers, some other quantity being taken as the unit; and even when the letters represent numbers, they are often called by the general name of *Quantities.* The words *Number* and *Quantity*, in the language of Algebra, are therefore synonymous.

4. While the figures of Arithmetic represent only *known* numbers, those of Algebra sometimes represent numbers *not known.* The letters at the commencement of the Alphabet are used to denote known numbers, as $a$, $b$, $c$; those at its close, to denote numbers unknown, as $x$, $y$, $z$.

5. Both Arithmetic and Algebra employ signs to connect the numbers, and to express the operations to which they are subjected, but Algebra employs such signs much more extensively.

6. + (named *plus*) is the sign of addition. Thus, $3+2$ means 3 added to 2, or 5.

7. − (*minus*) is the sign of subtraction, and means that the quantity following the sign is to be taken from that which precedes it. Thus, $5-2$ means 2 taken from 5, or 3.

8. When the sign + is prefixed to any number, and no other number stands before it to which it can be added (as $+ 7$), the number to which the sign is prefixed is called a *Positive Number.* When the sign – is similarly placed, as $(-7)$, the number to which it is prefixed is (along with the sign, called a *Negative Number;* that is, a number less than nothing. Even when such numbers or quantities are connected with other numbers by their signs, they may still be considered by themselves without reference to such other numbers, and then also described as positive or negative numbers.

All actual numbers, such as we usually speak of, and subject to the ordinary operations of Arithmetic, are positive numbers; but we cannot form an idea of negative numbers, unless relatively to positive numbers, of which they are the opposite. Thus, if $+5$ mean 5 Pounds of property, then $-5$ will mean 5 Pounds of debt; if $+3$ is used for 3 feet of height, then $-3$ stands for 3 feet of depth. When a number stands without either of these two signs, it is understood to be positive.

9. The signs + and – themselves are named *Positive* and *Negative Signs.*

10. ± (*plus or minus*) indicates that the second quantity is either to be added to, or subtracted from, the first.

EXERCISE A. What is the amount of each of the following expressions? $7+5$; $3+4+6$; $6+4+3$; $7-5$; $8+9-15$; $8-15+9$; $5-7$; $10 \pm 2$.

11. × (*into*) is the sign of multiplication. Thus, $3 \times 2$ means 3 times 2, or twice 3, or 6. Instead of this sign we often employ a point, as $3.2$; or, when letters are used, or figures and letters conjointly, we simply write them together; as $ab$ for $a \times b$; or $7bc$, meaning the product of the three numbers, $b$, $c$, and 7.

12. ÷ (*by*) is the sign of division, the dividend preceding it, and the divisor following. Thus, $21 \div 3$ means 21 divided by 3; that is, 7. The same thing is more frequently expressed by writing the divisor below the dividend in the form of a fraction, as $\frac{21}{3}$.

EXERCISE B. What is the simple value of each of the following expressions? $7 \times 9$; $9 \times 7$; $5 \times 8 \times 2$; $8.5.2$; $8 \div 2$; $\frac{12}{4}$; $\frac{9 \times 5}{3}$; and $9.5 \div 3$.

EXERCISE C. If the letter $a$ denote 10; $b$, 6; $c$, 2; $d$, 1;

and $e$, 0; what are the values of the following expressions? $b+c$; $b-c$; $b \pm c$; $a-e$; $a+b+c+d+e$; $a-b+c-d$; $a \times b$; $b.c$; $abc$; $15b$; $5ac$; $abcde$; $a \div c$; $\frac{b}{c}$; $\frac{d}{c}$; $\frac{3a}{b}$; $\frac{a+b}{c}$; $\frac{a-b}{c}$; $\frac{25b}{2a}$; and $\frac{ac-cd}{b}$.

13. $=$ (*equal to*) is the sign of equality. Thus we speak of $5+3=8$.

When a point is placed over the sign, we read the word *is* before it: thus we say $5+3 \doteq 8$, meaning $5+3$ *is equal to* 8.

14. : :: : is the sign of proportion. Thus, $a : b :: c : d$, is read, As $a$ is to $b$, so is $c$ to $d$.*

15. The sign ∴ signifies *therefore*, and ∵ signifies *because*.

16. When a number is the product of two or more numbers multiplied together, the latter numbers are called the *Factors* of the former. Thus, 7 and 3 are the factors of the number 21; $a$, $b$, and $c$ are the factors of the quantity $abc$.

EXERCISE D. What are the factors of the number 30?

17. A *prime Number*, or *prime Factor*, is that which cannot be resolved into other factors; as 2, 7, $a$, or $x+y$.

18. When the first of two or more factors is a figure, and the others letters, as $7ab$, the figure is called the *Co-efficient* of the letters. When no co-efficient is expressed, it is understood to be 1. Sometimes, also, a letter is taken as a co-efficient, especially when that letter represents a known number, and the letter, to which it is attached, an unknown number. Thus, in the expression $mx$, $m$ may be regarded as the co-efficient of $x$.

19. When all the factors are alike, as $ccc$, the product is called a *Power* of the single factor, and the latter is called a *Root* of the former. Thus, $bbbb$ is a power of $b$, and $d$ is a root of $ddd$. Thus, also, 27 is a power of 3, and 3 is a root of 27; for $27 \doteq 3.3.3$.

20. The *number* of equal factors is called the *Exponent* or *Index* of the power or root, and by that number the power or root is described. Thus, 16 is called the fourth power of 2; and 2 the fourth root of 16, the index being 4; for $16 \doteq 2.2.2.2$. Thus, also, $bbbbb$ is the fifth power of $b$, and the latter is the fifth root of the former, the index being 5.

21. Any power of a number is expressed briefly by writing the exponent over the number, as $4^2$, $a^6$,—mean-

---

* Many of these definitions belong to Arithmetic as well as to Algebra; but it is necessary to give a synopsis of the whole.

ing the second power of 4, and the sixth power of $a$. When no exponent is expressed, it is understood to be 1, any number being its own first power.

Exercise E. What are the numerical values of $8^2$, $12^2$, $2^3$, $3^3$, $5^3$, $2^4$, $3^4$, $2^5$, $2^6$, $2^7$, $15^2$, $10^3$, and $5^4$?

Exercise F. Taking $a$, $b$, $c$, $d$, and $e$ for 1, 2, 3, 4, and 5, what are the values of $c^4$, $b^8$, $d^3$, $e^5$, $a^4$, $2e^2$, $7b^3$, $a^2+b^2+c^2$, $b^2 \times c^2$, and $e^3-d^3$, expressed in numbers?

22. A root is expressed by prefixing the *Radical* Sign, $\surd$, with the exponent over it, as $\sqrt[5]{n}$,—meaning the fifth root of $n$.

Exercise G. What are the values of $\sqrt[2]{64}$, $\sqrt[3]{27}$, and $\sqrt[4]{16}$?

23. The *Square* of any number is its second power, and the *Cube* is its third power. In like manner the *Square Root* and the *Cube Root* are the second and third roots. When the radical sign occurs without an exponent expressed, the square root is meant, as $\sqrt{9}$.

Exercise H. What is the square root of 121, the cube root of 125, and $\sqrt{81}$?

24. Any single quantity, as $a$, or the combination of any number of quantities, by multiplication or division, as $b \times c$, or $2abc$, or $\frac{ab}{cd}$, is called a *Single Term*.

25. *Like Terms* are those made up of the same letters similarly combined, although their co-efficients may differ. Thus, $3a$ and $7a$ are like terms; $\frac{b}{c}$, $2\frac{b}{c}$, and $7\frac{b}{c}$, are also like terms.

26. *Unlike Terms* are those made up of different letters, or of the same letters so combined as to give different values. Thus $a$, $b$, $ab$, $\frac{a}{b}$, and $\frac{b}{a}$, are all unlike terms; but $ab$ and $ba$ are not unlike, for their values are necessarily the same.

27. A *Simple Quantity* is that which consists of a single term, as $a^3$ or $\sqrt{(3xy)}$.

28. A *Compound Quantity* consists of two or more terms connected together by the signs + or −; as $a+b-c$.

29. If a compound quantity consist of only two terms, it is called a *Binomial;* as $x-y$.

30. Terms following each other, and changing according to any given law, are called a *Series;* as, 2, 3, 4. When the series has no termination, it is called an *Infinite Series;* as 2, 4, 8, 16, 32, &c., *ad infinitum.*

31. A parenthesis or line, connecting the terms of a compound quantity, is called a *Vinculum;* as $\overline{m+n}$, or $(a+b-c)$, or $\left\{\frac{ax}{3}-\frac{by}{2}\right\}$. It indicates that the compound quantity so enclosed is to be taken as a whole, and no regard paid to the separate terms of which it consists.

32. The vinculum is sometimes required to connect not only the different parts of a compound quantity, but also the component parts of the same term, when these parts are all affected by one exponent of a power, or one radical sign; as $(7b)^3$, or $\sqrt{cd}$.

33. The line between the terms of a fraction serves as a vinculum both to the numerator and to the denominator, when compound quantities. The separating line has the same effect when division is expressed in the fractional form; as $\frac{a+b}{a-b}$.

NOTE. Additional definitions will be given under the heads to which they *particularly* belong.

---

# CHAPTER II.

## ADDITION.

*Addition*, in Algebra, is the uniting of various quantities into one simple quantity, or into a compound quantity in its simplest form.

In Arithmetic the addition of a number always increases that to which it is added. Not so in Algebra; for the addition of negative quantities neutralizes an equivalent of positive quantities, so that the addition of a negative quantity is the same as the subtraction of a positive quantity.

### PROBLEM I.

*To add simple Quantities together.*

CASE I. *When the quantities are like and with like signs.*

RULE. Add the co-efficients together by common Arithmetic, and set down the sum, followed by the common letter or letters, and preceded by the common sign, $+$ or $-$.

Thus, $4ab$ added to $3ab$ makes $7ab$, and $-6x$ added to $-4x$ makes $-10x$.

Demonstration. The rule is self-evident in the case of positive quantities: for, since 5 added to 3 makes 8, whether the 5, 3, and 8 are pounds, yards, gallons, or any other quantities or articles whatever, $5a$ added to $3a$ must be $8a$, $+5x$ added to $+3x$ must be $+8x$, and so on, by whatever letters the quantities may be expressed; and the same reasoning will be applicable when the quantities are more than two, and whatever the co-efficients may be.

In the case of negative quantities, the rule is almost equally evident, the negative sign merely expressing the kind or quality of the articles added together, and in no way affecting their number. Thus if £2 of property added to £4 of property, make £6 of property, so £2 of debt added to £4 of debt will make £6 of debt. In like manner, $-2$ added to $-4$ will make $-6$, and $-3a$ added to $-7a$ will make $-10a$; and so on, whatever the numbers and quantities.

Note. When there are many quantities to be added, they should be arranged in columns as in Arithmetic.

Exercise 1. Add together $3a$, $15a$, $7a$, and $9a$.
Ans. $34a$.

2. Add into one sum $+6b^2$, $+3b^2$, $+b^2$, and $+4b^2$.
Ans. $+14b^2$.

3. What is the sum of $-6c$, $-12c$, and $-11c$?
Ans. $-29c$.

4. Add together $-23a\frac{x}{y}$, $-67a\frac{x}{y}$, $-34a\frac{x}{y}$, and $-108a\frac{x}{y}$.

Ans. $-232a\frac{x}{y}$.

A. Sum each of the following groups of quantities:—

| | | | |
|---|---|---|---|
| $15ab$ | $+84\sqrt{x}$ | $-208x^2$ | $-377\times\frac{y}{z}$ |
| $7ab$ | $+29\sqrt{x}$ | $-36x^2$ | |
| $32ab$ | $+5\sqrt{x}$ | $-57x^2$ | $-800\times\frac{y}{z}$ |
| $ab$ | $+48\sqrt{x}$ | $-726x^2$ | |
| $55ab$ | $+16\sqrt{x}$ | $-244x^2$ | $-493\times\frac{y}{z}$. |
| $93ab$. | $+57\sqrt{x}$. | $-862x^2$. | |

Case 2. *When there are only two like quantities with unlike signs.*

*Rule.* Subtract the smaller co-efficient from the greater, and set down the remainder, preceded by the sign of the greater, and followed by the common letter or letters.

Thus, $+7x$ added to $-3x$ makes $+4x$, and $-7x$ added to $+3x$ makes $-4x$.

**Dem.** If we are required to add $-2a$ to $+5a$, it is the same thing as to add $-2a$ to $3a+2a$, since $5a \doteq 3a+2a$. But it follows, from the definition of negative quantities, that $-2a$ added to $+2a$, makes 0, since the two neutralize each other. $\therefore -2a$, added to $3a+2a$, makes just $3a$; or $-2a+5a \doteq 3a$. In like manner, if we are required to add $+2a$ to $-5a$; since $-5a \doteq -2a$ added to $-3a$, the sum of the whole will be $+2a-2a-3a=-3a$. And, in general, if we are required to add $-n$ to $+m+n$, the result will be $+m$; but if to add $+n$ to $-(m+n)$, or, which is the same thing, to $-m-n$, the result is $-m$.

**Exercise** 1. Add together $+10m$ and $-6m$.
Ans. $+4m$.

2. Add $-49n$ to $+25n$. Ans. $-24n$.

3. What is the sum of $+253\sqrt{(xy)}$ and $-741\sqrt{(xy)}$?
Ans. $-488\sqrt{(xy)}$.

B. Add the following, taking each column as a separate sum :—

| | | | |
|---|---|---|---|
| $+55c$ | $+12d^2$ | $-499\sqrt{(xy)}$ | $-54az$ |
| $-43c$. | $-96d^2$. | $+267\sqrt{(xy)}$. | $+300az$. |

**Case** 3. *When there are several like quantities with unlike signs.*

**Rule.** Add all the positive quantities into one sum, and all the negative into another, by Case 1. Then add these two sums together, by Case 2.

**Note.** When the quantities are numerous, we may, if we choose, place the positive quantities in one column and the negative in another.

**Example.** Add together $+7c$, $-5c$, $+c$, $+15c$, $-4c$, and $-10c$.

$$\begin{array}{rr} +\ 7c & -\ 5c \\ +\ 1c & -\ 4c \\ +15c & -10c \\ \hline \end{array}$$

$$+23c-19c=4c, \text{ Ans.}$$

**Exercise** 1. Add together $5a$, $-3a$, $-10a$, and $18a$.
Ans. $+10a$.

2. Collect into one sum, $-5bx$, $+7bx$, $-19bx$, $+3bx$, and $-12bx$. Ans. $-26bx$.

3. What is the sum of $+64m^2$, $-17m^2$, $+95m^2$, $+23m^2$, $-56m^2$, $-49m^2$, $-31m^2$, $+27m^2$, $-35m^2$?
Ans. $+21m^2$.

4. Add $19\sqrt{xy}$, $-45\sqrt{xy}$, $+38\sqrt{xy}$, $-39\sqrt{xy}$, $+27\sqrt{xy}$, together. Ans. 0.

C. Sum the following :—

| | | | |
|---|---|---|---|
| $+8y^2$ | $-540abc$ | $-25a^2x^2$ | $+430\sqrt{z}$ |
| $-7y^2$ | $+365abc$ | $-14a^2x^2$ | $+28\sqrt{z}$ |
| $+y^2$ | $+286abc$ | $+62a^2x^2$ | $+196\sqrt{z}$ |
| $-5y^2$ | $+194abc$ | $-49a^2x^2$ | $-85\sqrt{z}$ |
| $-9y^2$ | $-868abc$ | $+72a^2x^2$ | $-243\sqrt{z}$ |
| $+4y^2$. | $+444abc$. | $+50a^2x^2$. | $-9\sqrt{z}$. |

**Case 4.** *When all the quantities are unlike.*

**Rule.** All that can be done is to place them together, connected by their proper signs.

Thus, if we are required to add into one sum, $3a$, $-2b$, and $\sqrt{c}$, we cannot put that sum into a simpler form than $3a-2b+\sqrt{c}$.

**Exercise.** Add together $a$, $-b$, $+\sqrt{ab}$, and $-\frac{b}{a}$.

**Case 5.** *When there are several quantities, partly like and partly unlike.*

**Rule.** Add together the like quantities, as in Cases 1, 2, and 3, and connect the sums with each other (if more than one), and with the unlike quantities, as in Case 4.

**Example.** What is the sum of $+7a$, $-13b$, $+5x$, $+2a$, $-3y$, $-5x$, $-7y$, and $-2\sqrt{(xy)}$.

$$\begin{array}{lllll} & +7a-13b & +5x & -3y & \\ & +2a & -5x & -7y & -2\sqrt{(xy)} \\ \hline \text{Ans.} & +9a-13b & \ldots\ldots & -10y & -2\sqrt{(xy)}. \end{array}$$

**Exercise 1.** Add together $7a$, $4b$, $5c$, $6a$, $3b$, and $10a$.
Ans. $23a+7b+5c$.

2. Express in the simplest form $3a^2+9-7a+5\sqrt{b}-16+11a-13a^2-12\sqrt{(ab)}-23$.
Ans. $4a-10a^2+5\sqrt{b}-12\sqrt{(ab)}-30$.

3. Simplify the expression $17a^2-34b^2-51c^2+13a^2-26ab+13b^2-6a^2+12ac-6c^2+55b^2-110bc+55c^2$
Ans. $24a^2-26ab+34b^2+12ac-110bc-2c^2$.

D. Find the sums of the following columns separately :—

| | | | |
|---|---|---|---|
| $63x$ | $+12a$ | $-308\sqrt{(ax^3)}$ | $+17mx$. |
| $38y$ | $-19b$ | $+122\sqrt{(bx^3)}$ | $-19nx$. |
| $54z$ | $+48a$ | $+424\sqrt{(ax^3)}$ | $+34px$. |
| $20y$ | $+63b$ | $-852\sqrt{(bx^3)}$ | $+82px$ |
| $92z$ | $-15b$ | $+66\sqrt{(ax^3)}$ | $-54mx$. |
| $13z$. | $-8a$. | $-508\sqrt{(bx^3)}$. | $-80mx$. |

## PROBLEM II.

*To add compound Quantities together.*

**RULE.** Add the simple quantities of which they consist, separately.

**EXAMPLE 1.** Add together $3a-7b$, $5a+4b$, and $-2a-2b$.

$$\begin{array}{r} +3a-7b \\ +5a+4b \\ -2a-2b \\ \hline \end{array}$$

Ans. $6a-5b$.

**EXERCISE 1.** Add together $7a^2-3b^2$, $5a^2-12b^2$, and $4a^2-4b^2$. Ans. $16a^2-19b^2$.

2. Add together $3a-4\sqrt{(bc)}$, $-5a+8\sqrt{(bc)}$, $-10a-20\sqrt{(bc)}$, and $7a-6\sqrt{(bc)}$. Ans. $-5a-22\sqrt{(bc)}$.

3. Add into one sum, $4x^2+4xy+4y^2$, $7x^2-7xy-3y^2$, $-8x^2-8xy+5y^2$, $-3x^2+7xy-8y^2$, and $x^2+4xy+y^2$. Ans. $x^2-y^2$.

4. What is the sum of $x^3+3x^2y+3xy^2+y^3$, $5x^3-15x^2y+15xy^2-5y^3$, $10x^3+10y^3$, $17x^2y-17xy^2$, and $-13x^3+39x^2y-39xy^2+13y^3$? Ans. $3x^3+44x^2y-38xy^2+19y^3$.

A. Add the following separately:—

| | | |
|---|---|---|
| $28x-28y$ | $+33a-44b$ | $+\ 8\sqrt{x}-9\sqrt{y}$ |
| $15x-14y$ | $-60a-18b$ | $+12\sqrt{x}+7\sqrt{y}$ |
| $34x+50y$ | $+55a-27b$ | $-87\sqrt{x}-5\sqrt{y}$ |
| $17x-12y$ | $-83a-83b$ | $-34\sqrt{x}+8\sqrt{y}$ |
| $86x-42y$ | $-24a-36b$ | $+90\sqrt{x}+\ \sqrt{y}$ |
| $17x+74y$. | $+72a-\ 8b$. | $-38\sqrt{x}-6\sqrt{y}$. |

**NOTE 1.** It will sometimes be necessary to re-arrange the terms, in order that each may stand in its proper column.

**EXAMPLE 2.** Add together $3a+6x-7y$, $2a-7x+5b$, and $x+b+y$.

$$\begin{array}{lllll} 3a & & & +6x & -7y \\ 2a & +5b & & -7x & \\ & \ \ 1b & & +1x & +1y \\ \hline \end{array}$$

Ans. $5a+6b\ \ \dots\ \ -6y$.

**EXERCISE 5.** Find the sum of $a^2+b^2+c^2$, $-6ab-6ac-6bc$, $25a^2-10ab+b^2$, $a^2-8ac+16c^2$, and $-4b^2+6bc-9c^2$. Ans. $27a^2-16ab-2b^2-14ac+8c^2$.

6. Required, the amount of $7x^2-16+5y$, $24-\sqrt{(xy)}-15x^2+3xy$, and $3x^2-14+28y$.

Ans. $-5x^2+3xy-\sqrt{(xy)}+33y-6$.

7. Collect the following quantities into one sum:— $15a^2b+15ab^2-16ac^2+15b^2c$, $a^3-3ab^2-3ac^2-b^3+12c^3$, $-5a^3+20b^2c+15bc^2-6a^2c$, and $-12ab^2-12bc^2+16ac^2+24a^2b-8a^2c-15b^2c$.

Ans. $-4a^3+39a^2b-b^3-14a^2c-3ac^2+20b^2c+3bc^2+12c^3$.

B. Add together $a^2+b^2+c^2+d^2$, $a^2-2ab+b^2$, $a^2+10ac+25c^2$, $b^2-6bc+9c^2$, $4c^2-4cd+d^2$, $9a^2-24ad+16d^2$, $-16b^2+24bd-9d^2$, and $10ab+10ac+10bc+10ad+10bd+10cd$.

NOTE 2. When compound quantities are enclosed in vincula, they are often added together as if they were simple quantities, especially if several of them are alike. Powers and roots of compound quantities are also regarded as simple quantities, and added as such. So are fractions whose terms are compound.

EXAMPLE 3. What is the sum of $+3(a-b)$, $-5(a-b)$, and $+7(a-b)$?

$$\begin{array}{r} +3(a-b) \\ -5(a-b) \\ +7(a-b) \\ \hline \end{array}$$

Ans. $+5(a-b)$.

EXERCISE 8. Find the sum of $+7(a+x)$, $-13(a+x)$, $-22(a+x)$, $+55(a+x)$, $-36(a+x)$, $+18(a+x)$, and $-21(a+x)$. Ans. $-12(a+x)$.

9. Add together $3(a+b)$, $-5(a-b)$, $4(a-b)-6(a+b)$, $7(a+b)+7(a-b)$, and $(a-b)-(a+b)$.

Ans. $3(a+b)+7(a-b)$.

10. Add together $5(x-y)^2$, $-7(x-y)^2$, $+24(x-y)^2$, and $-13(x-y)^2$. Ans. $9(x-y)^2$.

11. Collect into one sum $4\sqrt{(x^2+y^2)}$, $5\sqrt{(x^2-y^2)}$, $-7\sqrt{(x^2-y^2)}$, $-6\sqrt{(x^2+y^2)}$, $+20\sqrt{(x^2+y^2)}$, and $-8\sqrt{(x^2-y^2)}$. Ans. $18\sqrt{(x^2+y^2)}-10\sqrt{(x^2-y^2)}$.

12. Add $369\,\dfrac{a^2+b^2}{a^2-b^2}$ to $-484\,\dfrac{a^2+b^2}{a^2-b^2}$.

Ans. $-115\,\dfrac{a^2+b^2}{a^2-b^2}$.

C. $+8(a^2-x^2)$, $-19(a^2-x^2)$, $+86(a^2-x^2)$, and $-27(a^2-x^2)$.

D. $+38(b+c)^3$, $-54(b+c)^3$, $-27(b+c)^3$, $-93(b+c)^3$, *and* $+804(b+c)^3$.

E. $+59 \times \frac{ax+by}{xy}$, $+94 \times \frac{ax+by}{xy}$, and $-35 \times \frac{ax+by}{xy}$.

F. $+12(ab+cd)$, $-22(ac+bd)$, $-5(ab+cd)$, and $+77(ac+bd)$.

# CHAPTER III.

## SUBTRACTION.

GENERAL RULE. Change the sign, or signs, of the subtrahend, and add it (thus changed) to the minuend.

DEM. When the quantity to be subtracted forms a part of that from which it is to be taken, we have merely to remove that part, the simplest idea of Subtraction being the removing of a part from the whole. Thus, $+n$ subtracted from $m+n$, gives $m$, and $-q$ subtracted from $p-q$, gives $p$.

But if the subtrahend is no part of the minuend, we may alter the form of the minuend (without changing its value) so that the subtrahend shall be equivalent to a part of it. Thus we know, by the rules of Addition, that $m = m + (n-n) = m+n-n$: and, if we are required to subtract $+n$ from $m$ ($m$ being any quantity, positive or negative), instead of taking $+n$ from $m$ let us take it from its equal, $m+n-n$, and there remains $m-n$; so that subtracting $+n$ is equivalent to adding $-n$. Again, if $-n$ is to be subtracted from $m$, we take it from its equal $m+n-n$, and there remains $m+n$; so that subtracting $-n$ is equivalent to adding $+n$.

The reason of the rule may be rendered more familiar by reverting to the illustrations given of positive and negative quantities in the definitions. Thus, regarding + as indicating property, and −, debt, we know that taking away property from any one is equivalent to adding just as much to his debt, or taking away so much of his debt is equivalent to giving him so much property.

NOTE. Instead of actually changing the signs of the subtrahend, it is better to write it anew with the signs altered. After some practice, however, it will be found sufficient not to change the signs at all, but merely to suppose them changed; and even that, after a little practice, will seldom be necessary.*

EXAMPLE. From $7a-16b+4c-5d+e-f$ subtract $a-9b+5c-8d+12e-13f$.

$$\begin{array}{l} +7a-16b+4c-5d+1e-1f \\ -1a+9b-5c+8d-12e+13f \\ \hline \end{array}$$

Ans. $6a-7b-c+3d-11e+12f$.

EXERCISE 1. From $36x$ take $15x$. Ans. $+21x$.

2. From $15x$ subtract $36x$. Ans. $-21x$.

3. From $-3y$ subtract $-17y$. Ans. $+14y$.

4. From $-17y$ subtract $-3y$. Ans. $-14y$.

A. From $5800x^2y^2$ subtract $2438x^2y^2$.

B. From $-34abc$ subtract $-100abc$.

C. From $+17my$ subtract $+83my$.

D. From $-88\sqrt{(x^5)}$ subtract $-29\sqrt{(x^5)}$.

5. From $m+12n$ take $15m+7n$. Ans. $-14m+5n$.

6. From $17m+3n$ take $-11m-12n$. Ans. $28m+15n$.

7. From $-6a^2-24b^2$ take $18a^2+9b^2$. Ans. $-24a^2-33b^2$.

8. From $-15\sqrt{(ab)}-7\sqrt{(ac)}$ subtract $-3\sqrt{(ab)}-12\sqrt{(ac)}$. Ans. $-12\sqrt{(ab)}+5\sqrt{(ac)}$.

9. Subtract $a^3-3a^2b+3ab^2-b^3$ from $a^3+6a^2b+12ab^2+8b^3$. Ans. $9a^2b+9ab^2+9b^3$.

10. Subtract $3a^5-9a^4+15a^3-24a^2+30a-45$ from $a^5+4a^4+20a^3+25a^2+20a-45$.

Ans. $-2a^5+13a^4+5a^3+49a^2-10a$.

11. Subtract $19a^2-24b^2+27ax+13y^2$ from $33a^2+7ac+46ax-12y^2-7z^2$.

Ans. $14a^2+24b^2+7ac+19ax-25y^2-7z^2$.

12. Subtract $4x^2y-7xy^2+5xz^2-24yz^2-xyz$ from $4x^2y-7x^2z+12y^2z-24yz^2+xyz$.

Ans. $7xy^2-5xz^2-7x^2z+2xyz+12y^2z$.

13. Take $12(a+x)^2-63(b-y)^2$ from $63(a+x)^2-12(b-y)^2$. Ans. $51(a+x)^2+51(b-y)^2$.

14. Take $19(a-b)-19(a+b)$ from $(a+b)+(a-b)$.

Ans. $20(a+b)-18(a-b)$.

E. From $81\sqrt{ay}-12\sqrt{bz}$ take $29\sqrt{ay}-20\sqrt{bz}$.

F. From $267mx^2-969ny^2$ take $607mx^2-242ny^2$.

G. From $9a-3b+2c-8d$ take $-3a+8b-7c+5d$.

---

* It will never be necessary in the case of like quantities with like signs when, the coefficient of the subtrahend is the smaller. Thus, if we are required to subtract $7a-3b$ from $10a-5b$, the remainder is evidently $3a-2b$, without any *change of signs.*

H. Take $584x - 362y + 607z$ from $372x - 584y - 43z$.

K. From $45(a+x) - 63(a+y)$ take $24(a+x) - 86(a+y)$.

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# CHAPTER IV.

## THE VINCULUM.

The vinculum, explained in Definitions 31 and 32, is of frequent and important employment in Algebra. *When it connects the factors of a simple quantity* its use is obvious, but will be rendered more familiar by a few practical examples.

EXERCISE 1. What are the simple numerical values of $3 \times 2^2$ and $(3 \times 2)^2$? Ans. 12 and 36.

A. Of $5 \times 4^2$ and $(5 \times 4)^2$?

2. Express the following in simple numbers: $4 \times \surd 9$, $4 \surd 9$, $\surd 4 \times 9$, $\sqrt{4 \times 9}$, and $\surd 4 \times \surd 9$.

Ans. 12, 12, 18, 6, and 6.

B. The same with the following: $16 \times \surd 25$, $16 \surd 25$, $\surd 16 \times 25$, $\sqrt{16 \times 25}$, $\surd(16 \times 25)$, and $\surd 16 \times \surd 25$.

3. Simplify the expressions $(4 \times 5 \times 6)^2$ and $\sqrt{6 \times 5 \times 3 \times 10}$.

Ans. 14400 and 30.

C. The same with $(2 \times 3 \times 4)^2$, $\sqrt{4 \times 9 \times 16}$, and $\sqrt[3]{3 \times 8 \times 9}$.

D. The same with $5 \times (2 \times 3)^3$, and $4 \surd (2 \times 8)$.

E. Taking $b = 2$, and $c = 3$, what are the numerical values of the expressions, $b^2c$, $bc^2$, and $(bc)^2$?

F. Putting $a$ for 36 and $d$ for 100, what are the values of $a \surd d$, $\surd a \times d$, $\sqrt{ad}$, $\surd (a \times d)$, and $\surd a \times \surd d$?

G. Having $a = 2$, $b = 4$, and $c = 8$, express the following quantities in figures: $(a \times b \times c)^2$, $(abc)^2$, $a^2b^2c^2$, $a(bc)^2$, $abc^2$, $ab^2c$, and $(ab)^2c$.

*When the vinculum connects the terms of a compound quantity* its use requires particular explanation, since no sign is so liable to be the source of mistakes, especially when it is to be removed from an expression in which it has previously a place, or when it is to be inserted in a quantity previously expressed without it. The following remarks, with the annexed exercises, will, it is hoped, be sufficient to explain clearly its use in such cases, and to guard against errors arising from its insertion or removal.

**Remark** 1. Any sign immediately preceding a vinculum, is applied to the whole quantity within the vinculum, and not to any one of its terms. Thus $-(a+b)$ is entirely different from $(-a+b)$: thus also $\sqrt{(9+7)} \doteq 4$; but $\sqrt{9}+7 \doteq 10$.

**Remark** 2. When no sign, within the vinculum, precedes the first term, that term is understood to have the sign $+$. Thus $-(a+b)$ is the same as $-(+a+b)$.

**Exercise** H. What are the values of $+(8+6)$, $+(8-6)$, $-(8+6)$, and $-(8-6)$?

K. What are the values of $8+2^2$, $(8+2)^2$, $8-2^3$, $(8-2)^3$, $\sqrt{4}+5$, and $\sqrt{(4+5)}$?

**Remark** 3. Since the sign $+$ prefixed to a vinculum containing a compound quantity, implies that the whole compound quantity is to be added, *the removal of a vinculum*, when preceded by $+$, will not require a change of any of the signs within the vinculum. Thus, $8+(6-2)$ is exactly equivalent to $8+6-2$.

**Remark** 4. When the sign $-$ is prefixed to a vinculum containing a compound quantity, since it implies that the whole compound quantity is to be subtracted, the removal of the vinculum, in this case, requires the signs of all the terms within the vinculum to be changed, as in Subtraction. Thus, $8-(6+2)$, or $8-(+6+2)$, becomes $8-6-2$; and $8-(6-2)$ becomes $8-6+2$.

**Note.** Remarks 3 and 4 are derived respectively from the rules for Addition and Subtraction.

**Exercise** L. Express the following quantities without the vinculum: $a+(b+c)$, $a+(b-c)$, $a-(b+c)$, and $a-(b-c)$.

M. Express the following compound quantities without the vinculum, but retaining all the terms separate: $10+(5+2)$, $10+(5-2)$, $10-(5+2)$, and $10-(5-2)$.

N. State the simple numerical value of each of the four compound quantities mentioned in the preceding exercise.

4. Simplify as much as possible the expression $(3x+2y+4z)+(x+2y+3z)$. Ans. $4x+4y+7z$.

5. The same with the expression $(3x+2y+4z)-(x+2y+3z)$. Ans. $2x+z$.

6. Also $(5a-7b+6c)+(5a-10b-20c)$.
Ans. $10a-17b-14c$.

7. Also $(5a-7b+6c)-(5a-10b-20c)$.
Ans. $+3b+26c$.

*O. Also* $(a+b)+(a-b)-(a-c)-(b+c)$.

P. Also $(7w + 35x - 54y - 12z) - (62w - 12x + 13y - 80z)$.

**Remark 5.** In like manner, *the introduction of a vinculum*, if it is to be preceded by a positive sign, requires that all the terms enclosed shall retain their signs without alteration; but if the vinculum is to be preceded by a negative sign, all the terms enclosed must have their signs changed.

**Note.** The last remark results directly from Remarks 3 and 4.

**Exercise Q.** Express the following compound quantities, with the last two terms of each enclosed in a vinculum preceded by the positive sign: $a+b+c$, $a+b-c$, and $a-b+c$.

R. Express each of the following compound quantities, with the two last terms enclosed in a vinculum preceded by the negative sign: $a-b-c$, $a-b+c$, and $a+b-c$.

# CHAPTER V.

## MULTIPLICATION.

### Problem I.

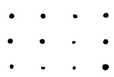
*To multiply two simple Quantities together.*

**Rule for the Letters and Co-efficients.** Multiply the two co-efficients together by common Arithmetic. After the product of the co-efficients write the letters of both factors together, in order.

Thus, the product of $a$ by $b$ is $ab$; that of $ax$ by $by$ is $abxy$; and that of $3x^2y$ by $2ab^2$ is $6ab^2x^2y$.

**Dem.** (1.) $a \times b \doteq ab$, these being synonymous expressions. See Ch. I, 11.

(2.) $ab \doteq ba$. Thus 3 times 4 is equal to 4 times 3. This is shown by placing the two factors in rows and cross rows. Thus, in the annexed diagram, we have either 4 rows of 3 in each row, or 3 rows of 4 in each row, according as we take the rows vertically or horizontally. But the whole number is the same whether we count



them the one way or the other. That is, $4 \times 3 \doteq 3 \times 4$; or, generally, $ab \doteq ba$.

(3.) $abc$ may either be regarded as $ab \times c$, or as $a \times bc$, since the two expressions are equivalent. That they *are* equivalent is not absolutely self-evident, although familiar use leads us to suppose so. A particular case will elucidate the principle most clearly. We are required to prove that $7 \times 15 \doteq 21 \times 5$, or that to multiply 7 by 15 is equivalent to multiplying 7 by 3, and the product by 5, which may be shown thus. The number $15 \doteq 5$ times 3, whatever the unit be: thus, 15 apples $\doteq$ 5 times (3 apples),* and 15 baskets of apples $\doteq$ 5 times (3 baskets of apples): 15 sevens, then, or 15 times 7, $\doteq$ 5 times (3 sevens) = 5 times $21 = 21 \times 5$.

(4.) $abc$ may also be regarded as $ac \times b$; for $abc \doteq a \times bc = a \times cb$, since $bc \doteq cb$, as shown above; and $a \times cb \doteq ac \times b$, as just proved.

(5.) If, therefore, one quantity is to be multiplied by the product of two others, or the product of two quantities to be multiplied by a third, or if the product of three separate quantities be required, in any of these cases, when the quantities are expressed by letters, we merely write the quantities together, and *that* in any order that may be most convenient. The same rule may evidently be extended, by the same process of reasoning, to any number of quantities.

(6.) From this too it follows that, if one of the quantities contain a numeral factor or co-efficient, it may be treated as a letter, and simply entered in the product; and that, if both quantities contain numeral factors, the product of these may be taken by itself and placed as a co-efficient to the product of the literal factors.

EXERCISE A. Multiply $c$ by $d$, and $mx$ by $ny$.

B. Multiply $3a$ by $bcd$, and $xz$ by $15y$.

C. Multiply $5a$ by $9b$, and $258a^2x^3$ by $37by^2$.

NOTE 1. *If the same letter occur in both factors without any exponent,* instead of writing it twice we may write its square, which is the same thing (Ch. I, 19, 21, and 23). Thus $a \times a \doteq a^2$, $ax \times bx \doteq abx^2$.

EXERCISE D. Multiply $35x$ by $23x$, $4my$ by $7ny$, and $13ab^2x$ by $12ac^2x$.

NOTE 2. *When two powers of the same letter occur in the two factors,* they are multiplied together by adding their exponents. Thus $b^2 \times b^3 \doteq b^5$.

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* The parenthesis is used here for the algebraical vinculum. Its use makes a nice distinction, which must be attended to.

DEM. $b^2 \doteq bb$, and $b^3 \doteq bbb$. $\therefore b^2 \times b^3 \doteq bb \times bbb = bbbbb = b^5$.

EXERCISE E. Multiply $a^5$ by $a^4$, and $13x^2$ by $4x^5$.

NOTE 3. In adding the exponents it must be kept in mind that, when no exponent is expressed, it is understood to be 1.

EXERCISE F. What is the product of $7c^2$ into $8c$, and of $15ab^2x^3$ into $16abx^4$?

RULE FOR THE SIGNS. Multiplying by a positive factor makes no change in the sign; but multiplying by a negative factor changes the sign.

*Or*,—The sign of the product will be + when the signs of the two factors are alike; but −, when they are different.

Thus $(+a) \times (+b) \doteq +ab$; $(-x) \times (+x) \doteq -x^2$; $(+y^3) \times (-y^4) \doteq -y^7$; and $(-5a^2x) \times (-4ax) \doteq +20a^3x^2$.

DEM. That multiplying by a positive number will not alter the sign of the multiplicand is clear; for taking $m$ times any number of things will not alter the character of those things. Thus, $m$ and $n$ being any numbers, $m$ times $n$ pounds will be pounds, $m$ times $n$ miles will be miles; so $m$ times any amount of property will be property, and $m$ times any amount of debt will be debt; $m$ times any height will be height, and $m$ times any depth will be depth. Therefore $m$ times $+n$ will be $+mn$, and $m$ times $-n$ will be $-mn$.

Or thus:—Multiplying by a positive quantity is equivalent to adding the multiplicand to itself as many times as there are units in the multiplier: and, therefore,

$$+5 \times 3 \doteq +5+5+5 = +15,$$
$$\text{and } -5 \times 3 \doteq -5-5-5 = -15,$$

by the first rule of Addition.

The multiplier, $m$ or 3, is, of course, the same as $+m$ or $+3$.

Having proved that − multiplied by + gives − (or rather quantities with those signs attached to them), it may be supposed that the next case is also proved, viz. that + multiplied by − gives −; and, in fact, in many standard works this has been assumed.* But, although $(+n) \times (-m)$ is the same in *result* as $(-n) \times (+m)$, yet in reality the processes are essentially different. We know readily what is meant by $m$ times a negative quantity, such as 7 times a given debt, but it requires explanation to show us what is meant by taking a positive quantity a negative number of times. To ascertain this clearly, let us see what we mean by taking any number so many times. Seven times a gallon is, of course, the same as seven gallons: so seven times a number is the same as seven such numbers added together: 7 times 10, for instance, just means 7 tens. A negative number of times, then, is explained in the same way: (minus 7)* times a gallon is just (minus 7) gallons; and $(-7)$ gallons means 7 gallons subtracted. So $(-7)$ tens signifies 7 tens subtracted; or any number taken $(-7)$ times is identical with 7 times that number subtracted. Thus $10 \times (-7)$ will be 70 subtracted, or $-70$; $8 \times (-5)$ will be forty subtracted, or $-40$; and, in general, $(+m) \times (-n)$ will be $mn$ subtracted, or $-mn$.

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* It will not do to assume this on the ground of what has already been proved, viz. that $ab$ is $= ba$, for that was proved only on the understanding that both $a$ and $b$ were positive quantities.

In like manner, $(-10) \times (-7)$ will be equivalent to 7 times $-10$ subtracted, that is to $-70$ subtracted, which makes $+70$; and, in general, $(-m) \times (-n)$ is the same as $-m \times n$ subtracted, or $+mn$.

In every case, therefore, multiplying any quantity by a negative number changes the sign of that quantity; and we have shown that multiplying by a positive quantity does not alter the sign. Consequently, when the signs are alike, the product is positive; when different, negative.

The subject may be still further elucidated to beginners by analogy. Let them be shown by instances that, when the multiplier and the multiplicand are both positive and the latter constant, the product always diminishes as we diminish the multiplier till both become 0 together; and, consequently, that if we diminish the multiplier still more, the multiplicand may be expected to diminish still more also; so that, when the multiplier becomes negative, the product becomes negative. On the other hand, when the multiplicand is negative and constant, and the multiplier positive, that the product increases as we diminish the multiplier. For instance, $-5 \times 4 \doteq -20$; but $-5 \times 3 \doteq -15$, which is greater than $-20$. If the multiplier is continually diminished until it vanishes, the product vanishes at the same time; and consequently, extending the analogy, we may expect that, when the multiplier becomes less than 0, the product will become greater than 0.

The latter method may not be regarded as strict demonstration; but it will show, at least, that there is *nothing*

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* The parenthesis is again used for the algebraical vinculum.

*strange* in the circumstance that multiplying a negative number by a negative number gives a positive number, but, on the contrary, that it is exactly what we might expect.

EXERCISE G. Multiply $-8a^5$ by $-9a^7$.

H. Multiply $+3abx^3$ by $+11acx^2$.

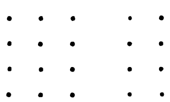
K. What is the product of $+17mn^4$ and $-2m^3n$?

## PROBLEM II.

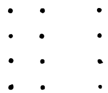
*To multiply a compound Quantity by a simple Quantity, or a simple Quantity by a compound.*

RULE. Multiply every term of the compound quantity separately by the simple quantity, and connect the products by their proper signs.

DEM. The rule rests on the fact that the product of the sum of two quantities by another quantity is equal to the sum of their separate products by that quantity; and that the product of the difference of two quantities by another quantity is equal to the difference of their separate products. Thus $(m+n) \times a \doteq am + an$, and $(m-n) \times a \doteq am - an$. The truth of that we may prove by placing $(m+n)$ or $(m-n)$ objects in a row, and taking $a$ such rows. Thus, if $m$ is 3; $n$, 2; and $a$, 4; our row, for the sum, will contain $(3+2)$ or 5, and we shall have 4 rows of 5 in each row, which we see at once is equal to 4 rows of 2 in each added to 4 of 3 in each. Therefore $(3+2) \times 4 \doteq (3 \times 4) + (2 \times 4)$.



So, for the difference, our row will contain $3-2$, or 1, and we shall have 4 rows of 1 in each, which is identical with 4 rows of 3, diminished by 4 rows of 2. Therefore $(3-2) \times 4 \doteq (3 \times 4) - (2 \times 4)$.



If $(m+n) \times a \doteq am + an$, and if $(m-n) \times a \doteq am - an$, it follows that $(m+n+p) \times a \doteq am + an + ap$, and $(m-n+p) \times a \doteq am - an + pn$; and, in the same manner, that $(m+n-p) \times a \doteq ma + na - pa$, and $(m-n-p) \times a \doteq ma - na - pa$; and, generally, that the same rule is applicable to any number of quantities connected in any way by the signs + and −.

If the multiplier is the compound quantity, and the multiplicand the simple quantity, the principle is the same.

**Example.** Multiply $4x^2 - x + 2$ by $3x^3$.

$$\begin{array}{r} 4x^2 - 1x + 2 \\ 3x^3 \\ \hline \text{Ans. } 12x^5 - 3x^4 + 6x^3. \end{array}$$

**Exercise 1.** Multiply $a^2 - 2b^2$ by $3xy$.

Ans. $3a^2xy - 6b^2xy$.

2. Multiply $7a - 5$ by $-8ab$, and $8x^2 - 9y^2$ by $+12x^3y^3$.

Answers: $-56a^2b + 40ab$, and $96x^5y^3 - 108x^3y^5$.

3. Multiply $-5mx$ by $6ax^3 - 8b^2x^2 - 20c^3x$.

Ans. $-30amx^4 + 40b^2mx^3 + 100c^3mx^2$.

A. Multiply $35a^3 + 12a^2x - 48ax^2 - 92x^3$ by $a^2x^2$.

B. Multiply $-369a^2b^5y$ by $612b^5y^7 - 356b^8y^9$.

## Problem III.

*To multiply one compound Quantity by another.*

**Rule.** Take either of the compound quantities as the multiplicand, and multiply that successively by the several terms of the other: then add the products together.

The *Demonstration* of the rule for this Problem is the same as that for the preceding. We take the multiplicand as *one* quantity ($a$), and we are required to show that $a \times (m \pm n) \doteq am \pm an$, which has been demonstrated under Problem II. If there are more than two terms in the multiplier, the principle is the same.

**Note 1.** The processes in this and the preceding Problem, when all the quantities are positive, correspond exactly to those with which we are familiar in common Arithmetic. When we multiply 43 by 2, we proceed as in Problem II; for $43 \times 2 \doteq (40 \times 2) + (3 \times 2)$; and when we multiply 39 by 25 we proceed as in Problem III; for $39 \times 25 \doteq (39 \times 20) + (39 \times 5)$.

**Note 2.** In multiplying it is usual to begin at the left hand, proceeding to the right.

**Example.** Multipy $a^2 + 2ax + x^2$ by $a^2 - 2ax + x^2$.

$$\begin{array}{llllll} a^2 + 2ax + x^2 & & & & & \\ a^2 - 2ax + x^2 & & & & & \\ \hline a^4 & + 2a^3x & + a^2x^2 & & & \\ & - 2a^3x & - 4a^2x^2 & - 2ax^3 & & \\ & & + a^2x^2 & + 2ax^3 & + x^4 & \\ \hline \text{Ans. } a^4 & \ldots & - 2a^2x^2 & \ldots & + x^4. & \end{array}$$

**Exer. 1.** Multiply $a + b$ by $a + b$. Ans. $a^2 + 2ab + b$ .

2. Multiply $3m - 2n$ by $2m - 3n$.
Ans. $6m^2 - 13mn + 6n^2$.

3. Multiply $2x + 2y$ by $2x - 2y$. Ans. $4x^2 - 4y^2$.

4. Multiply $5ax + 5ay$ by $4bx - 4by$.
Ans. $20abx^2 - 20aby^2$.

5. Multiply $x^2 + 2xy + y^2$ by $x - y$.
Ans. $x^3 + x^2y - xy^2 - y^3$.

6. Multiply $9a^2 + 6ab + 4b^2$ by $3a - 2b$.
Ans. $27a^3 - 8b^3$.

A. Multiply $3a - 5b$ by $7a - 6b$.

B. Multiply $34c - 35d$ by $35c - 34d$.

C. Multiply $7m - 9n$ by $9n - 7m$.

D. Multiply $a^2 + 6ab + 9b^2$ by $a^2 - 6ab + 9b^2$.

E. Multiply $a - 3a^2 + 9a^3$ into $1 + 3a$.

7. $-4m^2 + 10mn - 25n^2$ by $-4m^2 - 10mn - 25n^2$.
Ans. $16m^4 + 100m^2n^2 + 625n^4$.

8. $a^3 - 3a^2x + 3ax^2 - x^3$ by $a^3 + 3a^2x + 3ax^2 + x^3$.
Ans. $a^6 - 3a^4x^2 + 3a^2x^4 - x^6$.

9. $7abx - 9cdy$ by $12acx - 10bdy$.
Ans. $84a^2bcx^2 - 108ac^2dxy - 70ab^2dxy + 90bcd^2y^2$.

10. $8a^3 + 10a^2 + 12a + 14$ by $4a^3 - 5a^2 + 6a - 7$.
Ans. $32a^6 + 46a^4 - 68a^2 - 98$.

F. $27x^3 + 27x^2y + 9xy^2 + y^3$ by $27x^3 - 27x^2y + 9xy^2 - y^3$.

**Note** 3. When compound quantities are enclosed in vincula, they are often multiplied together as if they were simple quantities, especially when they are alike.

**Exercise** 11. Multiply $7(a - b)$ by $15(a - b)$.
Ans. $105(a - b)^2$.

12. Multiply $-45a^2b(m + n)$ by $+36a^2c(m + n)^2$.
Ans. $-1620a^4bc(m + n)^3$.

13. Multiply $+5\frac{a+b}{a-b}$ by $-3\frac{a+b}{a-b}$. Ans. $-15\left(\frac{a+b}{a-b}\right)^2$.

14. Multiply $-13(x + y)^2$ by $-8(x + y)^3$.
Ans. $104(x + y)^5$.

G. Multiply $54(x - y)$ into $75(x - y)$.

H. Multiply $-43(y + z)^3$ into $-26(y + z)^5$.

K. Multiply $5(a + b) + 7(a - b)$ by $5(a + b) - 7(a - b)$.

## Problem IV.

*To find the Product of any Number of Factors, whether simple or compound.*

**Rule.** Multiply the first by the second; then that product by the third; and so on.

NOTE 1. We may take the factors in any order that may be most convenient.

NOTE 2. In the case of a considerable number of simple factors, if the number of negative factors is even, the product will be positive; if odd, negative.

EXERCISE 1. Multiply $2a$, $3b$, $5c$, and $8d$, continually together. Ans. $240abcd$.

2. Express, in its simplest form, the product $(-15a^2x^4) \times (-12ax^2) \times (-7)$. Ans. $-1260a^3x^6$.

A. Multiply together $+7ax$, $-5by$, and $-4cz$.

B. $+3a^2x$, $-5b^2y$, $-7bx^2$, and $+4ay^2$.

C. $-14m^2nx^2y$, $+8mn^2xyz$, $-5mnxy^2z$, and $-6m^2n^2z^2$.

3. $a^2+b^2$, $a+b$, and $a-b$. Ans. $a^4-b^4$.

4. $a+b-c$, $a-b+c$, and $-a+b+c$.

Ans. $-a^3+a^2b+ab^2-b^3-2abc+a^2c+ac^2+b^2c$
$+bc^2-c^3$.

5. $x-3$, $x+3$, $x-5$, and $x+5$. Ans. $x^4-34x^2+225$.

6. Also, $x+y$, $x-y$, $x^2+xy+y^2$, and $x^2-xy+y^2$.
Ans. $x^6-y^6$.

D. Find the continued product of $a+b$, $a+c$, and $a+d$.

E. Of $x-2$, $x-3$, $x-4$, and $x-5$.

F. Of $x+y+z$, $x+y-z$, $x-y+z$, and $-x+y+z$.

G. Of $a^2+7a+49$, $a^2-7a+49$, $a+7$, and $a-7$.

# CHAPTER VI.

## DIVISION.

### PROBLEM I.

*To divide one simple Quantity by another.*

RULE. Cancel all the factors common to the dividend and divisor. Then, if all the factors of the divisor disappear, the quotient will be an integer consisting of the uncancelled factors of the dividend; but, if not, the quotient will be a fraction, having for its numerator the uncancelled factors of the dividend, and for its denominator those of the divisor.

*The rule for the sign* is similar to that for Multiplication, viz.—Dividing by a positive quantity makes no change in the sign of the dividend; but dividing by a negative quantity changes the sign. *Or*,—If the signs of the divisor and dividend are alike, that of the quotient will be $+$ ; but if unlike, it will be $-$.

**Dem.** Division is the reverse of Multiplication. In Multiplication we are required to take a certain quantity (or number) a given number of times. In Division we are required to ascertain how many times a given number is taken to make another given number. That other number, then (the dividend), we suppose to be the product of the other two, one of which (the divisor) is given, the other (the quotient) being to be found. But if the dividend be the product of the divisor and quotient, it must be made up of the factors of the quotient and divisor together. Consequently, in the case of simple quantities, if we desire to find the quotient, we have merely to separate the factors of the dividend, and take away from them those of the divisor, when those remaining will be the factors of the quotient. For instance, if we are required to divide $abcd$ by $ab$, what we mean is, to find by what number $ab$ must be multiplied to make $abcd$: now, whatever that number be, we know from multiplication that, when it is found, its factors along with those of $ab$ must just make up $abcd$, consequently the number must be $cd$, or, in other words, must be the product of all the other factors of $abcd$.

This applies to the case in which all the factors of the divisor are found explicitly in the dividend. If they are not, another process is required. Let us take, first, the case of $a \div b$, in which $b$ has no factor contained in $a$. In that case the quotient is the fraction $\frac{a}{b}$; for we shall afterwards prove, when we come to the head of Fractions, that $\frac{a}{b} \times b \doteq a$.

Next, to divide $ac$ by $bc$, $a$ and $b$ being understood to have no common factor. Since $ac \doteq \frac{a}{b} \times bc$, $ac \div bc \doteq \left(\frac{a}{b} \times bc\right) \div bc = \frac{a}{b}$. Or thus:—since we have proved that $a \div b \doteq \frac{a}{b}$, $a$ and $b$ being any numbers whatever, it follows that $ac \div bc \doteq \frac{ac}{bc}$; and this fraction reduced to its lowest terms, is $\frac{a}{b}$. By either way the common factor has disappeared.

In the preceding demonstration, however, the subject of Fractions has been anticipated. The proof is therefore incomplete in the present stage of our progress, and must be assumed, without further demonstration, till we have attained a more advanced position.

The rule for the sign follows evidently from that of Multiplication.

NOTE 1. It is usual, in the case of simple quantities, to write the divisor below the dividend in the form of a fraction, in order that the common factors may be seen more readily by standing in contiguity.

The common factors *of the co-efficients* will often be seen at a glance; but, if not, they are all cancelled at once by dividing both co-efficients by their greatest common measure, as found by Arithmetic. The common factors *among the letters* will be seen by mere inspection, when different powers of the same letter do not enter into the terms.

EXAMPLE 1. Divide $63abx^2y$ by $-9ax^2$.

$$\frac{+63abx^2y}{-9ax^2} \doteq -7by, \text{ Ans.}$$

EXERCISE A. Divide $6ab$ by $2b$, $125xy$ by $5x$, $+16m^3n$ by $-8m^3$, and $-84ab^2c$ by $-12b^2$.

EXAMPLE 2. Divide $-40mnyy$ by $-48nxy$.

$$\frac{-40mnyy}{-48nxy} \doteq +\frac{5my}{6x}.$$

EXERCISE B. Divide $7ab$ by $7ac$, $+54ax^2y$ by $+9bx^2y$, $+108ab^3c^2$ by $-84ab^3d^2$, and $-78mx^3$ by $-64my^3$.

NOTE 2. But when the divisor and dividend contain different powers of the same quantity, cancel that power which has the smaller exponent, and, at the same time, diminish the exponent of the higher power by that of the lower.

EXAMPLE 3. Divide $-99a^5b^4c^3$ by $+11a^2b^2c^2$.

$$\frac{-99a^5b^4c^3}{+11a^2b^2c^2} \doteq -9a^3b^2c.$$

EXERCISE C. Divide $104a^4b^5$ by $13a^3b^3$, $35x^6y^4z^2$ by $-7x^2y^4z^2$, $-90mn^2x^5y^5$ by $+mnxy^3$, and $-1728a^3x^4y^2$ by $-288axy$.

EXAMPLE 4. Divide $-24ab^3c^5$ by $-36a^5b^3c$.

$$\frac{-24ab^3c^5}{-36a^5b^3c} \doteq +\frac{2c^4}{3a^4}$$

*EXERCISE* D. Divide $1694a^2x^6y^4z^2$ by $616ax^2y^4z^6$,

$+114mn^3x^{10}y$ by $-116m^2nxy^{10}$, $-55ab^2c^3d^4$ by $+88a^4b^3c^2d$, and $-15a^{10}b^5$ by $-225a^5b^5$.

**Exercise E.** Divide $+1728a^5b^4c^3$ by $-144a^3b^3c^3$, $-64x^4y^4z^4$ by $-56xy^2z^3$, and $-1080xy^3$ by $+45x^3y$.

**Note 3.** If the letters of the divisor are all cancelled, but not the co-efficient, we may give the letters of the quotient in an integral form, and the co-efficient in a fractional form; and if the co-efficient of the divisor is cancelled, and not all its letters, we may give the co-efficient of the quotient as an integer and the letters as a fraction. Thus, $3a^5 \div 4a^2 \doteq \frac{3}{4}a^3$, and $12xy \div 3yz \doteq 4\frac{x}{z}$.

**Exercise F.** Divide $42abc^2$ by $36bc$, and $-108x^2y^3$ by $132xy$.

G. Divide $81ab^2x$ by $-27a^2bx$, and $-104my^6$ by $-8ny^3$.

## PROBLEM II.

*To divide a compound Quantity by a simple Quantity.*

**Rule.** Divide the terms of the dividend separately, and connect the quotients by the proper signs.

The *Demonstration* of the rule rests on the principle that $(a \pm b) \div m \doteq \frac{a}{m} \pm \frac{b}{m}$; and *that* we know, because $\left(\frac{a}{m} \pm \frac{b}{m}\right) \times m \doteq a \pm b$.

**Note.** The divisor may either be written below the dividend, and the process finished in the same form as in Problem I.; or the divisor may be written to the left of the dividend, and the quotient below, as in short division in common Arithmetic. The following example will exhibit both ways:—

**Example.** Divide $24a^3x^3 - 18a^5x - 10a^2x^4$ by $-6a^2x^2$.

$$\frac{24a^3x^3 - 18a^5x - 10a^2x^4}{-6a^2x^2} = -4ax + 3\frac{a^3}{x} + \frac{5}{3}x^2.$$

*Or thus:* $$-6a^2x^2\overline{)\,24a^3x^3 - 18a^5x - 10a^2x^4}$$

$$\text{Ans. } -4ax + 3\frac{a^3}{x} + \frac{5}{3}x^2.$$

**Exercise A.** Divide $a^3x^2 - 5abx^3 + 6ax^4$ by $ax^2$.

B. How often is $-8xy$ contained in $-16mx^3y + 24nx^2y^2 + 56pxy^3$?

C. Divide $30m^2n^3x^4-50m^3n^2x^4$ by $8m^2n^2x^2$.

D. Divide $56a^2y-72b^2y$ by $-8aby^2$.

E. Divide $35a^4b-84a^3b^2-105a^2b^3$ by $7a^2b$.

F. Divide $96m^2a^2-9m^2n^2$ by $-12mn$.

## PROBLEM III.

*To divide one compound Quantity by another.*

RULE. Arrange both the divisor and the dividend according to the powers of some one letter contained in them: then divide the first term of the dividend by the first term of the divisor, for the first term of the quotient. Multiply the whole divisor by the term thus found. Subtract the product from the corresponding terms of the dividend: bring down to the remainder as many other terms of the dividend as may be required; and repeat the operation till all the terms are brought down.

NOTE 1. The answers should all be proved by multiplication.

EXAMPLE. Divide $a^3-6a^2b+12ab^2-8b^3$ by $a-2b$.

$$\begin{array}{l}
a-2b)a^3-6a^2b+12ab^2-8b^3(a^2-4ab+4b^2\text{, Ans.}\\
\qquad\quad\ \underline{a^3-2a^2b\qquad\qquad}\\
\qquad\qquad\quad -4a^2b+12ab^2\\
\qquad\qquad\quad \underline{-4a^2b+\ \ 8ab^2\qquad}\\
\qquad\qquad\qquad\qquad +\ \ 4ab^2-8b^3\\
\qquad\qquad\qquad\qquad +\ \ 4ab^2-8b^3.
\end{array}$$

DEM. The rule depends upon the same principle as that of Problem II., viz. that the whole of any quantity divided by another quantity will be equal to the sum of the quotients arising from dividing the several parts of the former by the latter; for the dividend is made up of the several products which make their appearance in the course of the operation, and of the remainder, added together. Thus, in the Example, the dividend, $a^3-6a^2b+12ab-8b^3$, is equivalent to

$$\left\{\begin{array}{l}
a^3-2a^2b\\
\qquad -4a^2b+8ab^2\\
\qquad\qquad\qquad +4ab^2-8b^3.
\end{array}\right\}$$

Now the first line of this, divided by $a-2b$, gives $a^2$; the second gives $-4ab$; and the third, $+4b^2$. Therefore the whole quantity, divided by $a-2b$, is $a^2-4ab+4b^2$.

EXERCISE 1. Divide $a^2-2ab+b^2$ by $a-b$. Ans. $a-b$.

2. Divide $6a^2+5ab-6b^2$ by $2a+3b$. Ans. $3a-2b$.

3. Divide $6m^3 - 13m^2 + 6m$ by $3m - 2$. Ans. $2m^2 - 3m$.

4. Divide $am - an + bm - bn$ by $m - n$. Ans. $a + b$.

5. Divide $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$ by $x^2 - 2xy + y^2$. Ans. $x^3 - 3x^2y + 3xy^2 - y^3$.

6. Divide $6xy^3 + 6x^2y^2 - 8x^3y - 8x^4$ by $2x^2 + 2xy$. Ans. $-4x^2 + 3y^2$.

7. Divide $12a^4 - 192$ by $3a - 6$. Ans. $4a^3 + 8a^2 + 16a + 32$.

8. Divide $64z^6 - 729$ by $4z^2 - 9$. Ans. $16z^4 + 36z^2 + 81$.

9. Divide $10a^5 - 21a^4b + 27a^3b^2 - 34a^2b^3 + 11ab^4 + 5b^5$ by $2a^3 - 3a^2b + 4ab^2 - 5b^3$. Ans. $5a^2 - 3ab - b^2$.

A. Divide $b^2 + 4by + 4y^2$ by $b + 2y$.

B. Divide $8a^2 - 2ab - 15b^2$ by $4a + 5b$.

C. Divide $-16x^4 + 9y^4$ by $4x^2 - 3y^2$.

D. Divide $a^3 - b^3$ by $a - b$.

E. Divide $25a^4 + 50a^2 - 100a + 200$ by $5a^2 - 10a + 10$.

F. Divide $z^5 - 3125$ by $z - 5$.

NOTE 2. If there is a remainder after the process is completed, write the remainder as the numerator of a fraction, having the divisor for its denominator, and attach this to the quotient with the sign + or −. The former of these signs will usually be employed when the first term of the remainder is positive; but the latter, when negative.

DEM. The remainder is merely a certain part of the dividend still remaining to be divided, and not containing the divisor at all; and since it is therefore incapable of being divided so as to produce an integral quantity, the only resource is to annex it in a fractional form to the integral part of the quotient, for, as we shall afterwards demonstrate, $\frac{a}{b} \doteq a \div b$.

EXERCISE 10. Divide $x^4 - 2x^2y^2 + 2y^4$ by $x^2 - y^2$.

Ans. $x^2 - y^2 + \frac{y^4}{x^2 - y^2}$.

11. Divide $4a^2 - 10b^2$ by $2a - 3b$.

Ans. $2a + 3b - \frac{b^2}{2a - 3b}$.

12. Divide $8z^4 - 12z^2 - 3z$ by $2z^3 - z^2 - 3z$.

Ans. $4z + 2 + \frac{2z^2 + 3z}{2z^3 - z^2 - 3z}$.

G. Divide $10a^2 + 49ab - 10b^2$ by $2a + 11b$.

H. Divide $25x^4 - 90z^4$ by $5x^2 - 9z^2$.

K. Divide $84a^4 + 99a^3 - 7a^2 + 90a - 81$ by $12a^2 - 3a + 5$.

L. Divide $8x^3 - 4x^2y - xy^2$ by $4x^2 + 4xy + y^2$.

M. Divide $30y^3 - 67y^2 + 80y$ by $5y^2 - 7y + 6$.

NOTE 3. When the negative sign is attached to the fraction, the numerator of which is a compound quantity, we must change all the signs of the numerator, according to Remark 5 of Chapter IV. See also Def. 33, Ch. I.

EXERCISE 13. Divide $a^3 - 3a^2b + 2ab^2 - 2b^3$ by $a^2 - 2ab + b^2$.

Ans. $a - b - \frac{ab^2 + b^3}{a^2 - 2ab + b^2}$.

14. Divide $a^4 + 2a^3b + 2a^2b^2 + ab^3$ by $a^2 + 2ab + b^2$.

Ans. $a^2 + b^2 - \frac{ab^3 + b^4}{a^2 + 2ab + b^2}$.

15. Divide $32a^5 - 240a^4b + 720a^3b^2 - 1100a^2b^3 + 800ab^4 - 240b^5$ by $8a^3 - 36a^2b + 54ab^2 - 27b^3$.

Ans. $+ 4a^2 - 12ab + 9b^2 - \frac{20a^2b^3 + 10ab^4 - 3b^5}{8a^3 - 36a^2b + 54ab^2 - 27b^3}$.

N. Divide $x^5 - 4x^3 - 20x$ by $x^3 - 2x^2 + 3x - 4$.

O. Divide $3a^4 - a^3b - 14a^2b^2$ by $3a^3 + 5a^2b - 2ab^2 + b^3$.

NOTE 4. Sometimes, instead of the remainder being put down in a fractional form, the process of division is carried further, and the quotient continued in the form of a series; and, when a sufficient number of terms have been found to show the *law* of the series, as it is called, the quotient is concluded with "etc." Sometimes, also, a *simple* quantity is divided by a compound quantity in the same manner.

EXERCISE 16. Divide $1 + a$ by $1 - a$.

Ans. $1 + 2a + 2a^2 + 2a^3 +$ etc.

17. Divide $1 - a$ by $1 + a$. Ans. $1 - 2a + 2a^2 - 2a^3 +$ etc.

18. Divide 1 by $1 + x$. Ans. $1 - x + x^2 - x^3 +$ etc.

19. Divide 1 by $1 - x$. Ans. $1 + x + x^2 + x^3 +$ etc.

20. Divide $9 - 18x^2$ by $9 + 18x^2$.

Ans. $1 - 4x^2 + 8x^4 - 16x^6 +$ etc.

21. Divide 1 by $1 - 2a + a^2$.

Ans. $1 + 2a + 3a^2 + 4a^3 +$ etc.

P. Divide $2 + 6x$ by $1 - 3x$.

Q. Divide 10 by $1 - 5y$.

# CHAPTER VII.

## INVOLUTION.

*Involution* is the method of raising any given quantity to *any given power.*

## PROBLEM I.

### *To find any proposed Power of a simple Quantity.*

RULE. Multiply the exponent of every *literal factor* by the exponent of the proposed power, and to the result prefix the same power of the given *co-efficient*, as found by Arithmetic.

When the *sign* of the given quantity is +, that of every power will be +; and when the sign of the given quantity is −, that of its even powers will be +, and of its odd powers −.

NOTE. It will be recollected, that when no exponent is expressed, it is understood to be 1.

DEM. Since $(ab)^3 \doteq ab.ab.ab = aaabbb = a^3b^3$, and since the same reasoning may be applied to any power, or to any number of factors, it follows that—to raise a number composed of several factors to any power, we raise each factor separately to that power.

Now, if one of the factors is a particular number, as 5, the same principle holds good: that is—we raise the given number separately to the required power, and the literal factors separately to the same power. Hence the rule for the *co-efficient.*

Next let any of the factors of the given quantity be itself a power, as $a^2$, and let it be required to raise that to some given power, say the third. Since $a^2 \doteq aa$, $(a^2)^3 \doteq (aa)^3 = aa \times aa \times aa = aaaaaa = a^6$. But 6, the exponent of the power found, is the same as the number of times that $a$ is expressed in $aaaaaa$, which, again, was derived from the three factors, each containing $a$ twice, making 3 times 2, or, in other words, the exponent of the given literal factor multiplied by the exponent of the proposed power.

The rule for *the sign* is evident; for, if the given quantity have the sign +, it will continue to have + through any number of multiplications into itself. But if the given quantity have the sign −, it becomes + for the second power (since − into − gives +), then − for the third power (since + into − gives −), and so on.

EXERCISE 1. What is the cube, or third power, of $x^4$? Ans. $x^{12}$.

2. The square, or second power, of $7a^5b^3$? Ans. $49a^{10}b^6$.
3. Find the fourth power of $-5xy^4$. Ans. $+625x^4y^{16}$.
4. The seventh power of $-2ab^2c^3$. Ans. $-128a^7b^{14}c^{21}$.

A. The square of $283a^8$.
B. The cube of $25x^5y$.
C. The second power of $-29c^6$.
D. The third power of $-16y^2$.
E. The fifth power of $-4ab^3$.
F. The eighth power of $-2a^2y^4$.

## PROBLEM II.

*To find any proposed Power of a compound Quantity.*

RULE. Raise it to that power by actual multiplication. The number of factors will, of course, be the same as the exponent of the proposed power (See Chap. I., Def. 20); and, consequently, the number of operations will be one fewer.

EXERCISE 1. Find the square of $a+b$.
Ans. $a^2+2ab+b^2$.

2. The cube of $a-b$. Ans. $a^3-3a^2b+3ab^2-b^3$.

3. The second power of $7x-9y$.
Ans. $49x^2-126xy+81y^2$.

4. The fourth power of $5x-1$.
Ans. $625x^4-500x^3+150x^2-20x+1$.

5. The fifth power of $2m-3n$. Ans. $32m^5-240m^4n + 720m^3n^2-1080m^2n^3+810mn^4-243n^5$.

6. The seventh power of $x-y$. Ans. $x^7-7x^6y+21x^5y^2 - 35x^4y^3+35x^3y^4-21x^2y^5+7xy^6-y^7$.

7. The second power of $x-2y+3z$.
Ans. $x^2-4xy+6xz+4y^2-12yz+9z^2$.

8. The third power of $4a^2+5a$.
Ans. $64a^6+240a^5+300a^4+125a^3$.

A. Find the square of $5a-6b$.
B. The square of $7x-8y+9z$.
C. The cube of $a^2-b^2$.
D. The fourth power of $3c-4d$.
E. The third power of $x-y-z$.

NOTE. The learner should retain in memory the expressions for the squares of $a+b$ and $a-b$, as well as that for the product of $a+b$ by $a-b$, viz.:—

$$(a+b)^2=a^2+2ab+b^2.$$
$$(a-b)^2=a^2-2ab+b^2.$$
$$(a+b)\times(a-b)=a^2-b^2.$$

*He should also* be able to apply these as formulæ for cases in which $a$ and $b$ have particular values. In that way the powers and products in the following exercises should be found without actual multiplication, being repeated or written without any intermediate operation.

Exercise F. Give, at length, the expression for $(x+y)^2$.

G. For $(x+2)^2$.
H. For $(2+x)^2$.
I. For $(x+5)^2$.
J. For $(x+2y)^2$.
K. For $(2x+y)^2$.
L. For $(x+4y)^2$.
M. For $(3m+3n)^2$.
N. For $(c-d)^2$.
O. For $(x-1)^2$.
P. For $(3-y)^2$.
Q. For $(y-3z)^2$.
R. For $(3x-2y)^2$.
S. For $(z-7)^2$.
T. For $(2b+8c)^2$.
U. For $(3c-4d)^2$.
V. For $(x+y)\times(x-y)$.
W. For $(x+2)\times(x-2)$.
X. For $(x+12y)\times(x-12y)$.
Y. For $(20+z)\times(20-z)$.
Z. For $51\times 49$.

# CHAPTER VIII.

## EVOLUTION.

*Evolution* is the reverse of Involution. It is the method of finding any proposed root of any given quantity.

### Problem I.

*To find any proposed Root of any given simple Quantity.*

Rule. Divide the exponent of every *literal factor* in the given quantity, by the exponent of the proposed root, and prefix the proposed root of the given *co-efficient*, as found by Arithmetic.

*When the sign of the given quantity is* $-$, the extraction of any even root is impossible, and any odd root will have the sign $-$. *When the sign of the given quantity is* $+$, the sign of every odd root is $+$, and the sign of every even root either $+$ or $-$.

Thus $\sqrt{a^6}$ is $a^3$; $\sqrt{(+64x^2y^4)}$ is $+8xy^2$; $\sqrt{(-64x^2y^4)}$ has no existence, or is what is called an impossible quantity; $\sqrt[3]{+8a^6}$ is $+2a^2$; $\sqrt[3]{-125x^3}$ is $-5x$; $\sqrt[4]{(16z^{12})}$ is $2z^3$.

**Dem.** Since evolution is merely a reversing of the process of involution, and since, in the latter, we multiply the exponent of the given quantity by that of the power to which it is to be raised, we must, consequently, in evolution divide the given exponent by that of the root proposed to be extracted.

The method for the co-efficient belongs to common Arithmetic, but will be explained in the demonstration of the following problem.

Exercise 1. What is the square root of $9a^2$? Ans. $\pm 3a$.
2. The cube root of $64x^3$? Ans. $4x$.
3. The third root of $-125x^{12}y^6$? Ans. $-5x^4y^2$.
4. The fourth root of $81x^{12}y^8$? Ans. $\pm 3x^3y^2$.
A. Find the square root of $81a^2b^2$.
B. The square root of $+18496x^4y^6$.
C. The square root of $+27{\cdot}04a^8b^{10}c^{12}$.
D. The square root of $-9604x^{20}$.
E. The cube root of $+27x^{12}$.
F. The third root of $-216x^3y^9z^{12}$.
G. The fourth root of $4096c^{12}z^8$.

Note 2. When the exponent of the root proposed does not measure the exponent of any given literal factor, the result, as far as regards that factor, may either be given with the radical sign of the proposed root attached to it, or with the exponent in a fractional form. Thus $\sqrt{}(49a^4b)$ is either $\pm 7a^2\sqrt{b}$ or $\pm 7a^2b^{\frac{1}{2}}$. So also $\sqrt[3]{}(-512x^2)$ is expressed either as $-8\sqrt[3]{}(x^2)$ or as $-8x^{\frac{2}{3}}$. The fractional exponent is found in the same way as the integral,—that is, by dividing the given exponent by that of the proposed root.

Exercise H. Find the square root of $144x^2y$.
K. The cube root of $+343a^3b$.
L. The cube root of $-1000x^2y^6$.
M. The square root of $169ab^3$.
N. The fourth root of $-625x^5$.
O. The fifth root of $-243m^2n^3$.

## Problem II.

*To find the Square Root of any given compound Quantity.*

Rule.—1. Arrange the terms of the given quantity according to the powers of one of the letters. Find the square root of the first term by the preceding problem, and *place it as the first* term of the required root.

2. Place the square of the root thus found under the first term of the given quantity, to which, of course, it will be equal. Bring down the two next terms of the given quantity for a dividend, and twice the first term of the root for a divisor.

3. Divide the first term of the said dividend by the first term of the divisor. The quotient will be the second term of the root, to which it must be annexed with its proper sign, as also to the divisor.

4. Multiply the divisor, thus enlarged, by the second term of the root, and subtract the product from the last dividend.

5. To the remainder bring down two more terms of the given quantity, and use that as a new dividend. The new divisor will be the double of that part of the root already found.

6. Proceed with the new divisor and dividend as with the previous; and so on until there is no remainder, and no more terms to be brought down, or as far as may be thought necessary if the remainder do not disappear.

Note 1. Instead of doubling the root on each occasion for a divisor, add the term last found to the previous divisor, which amounts to the same thing in the result.

Example. Extract the square root of $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$.

$$\begin{array}{r|l}
a^2 & a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 (a^2 - 2ab + b^2. \\
a^2 & a^4 \\
\hline
2a^2 - 2ab & -4a^3b + 6a^2b^2 \\
2ab & -4a^3b + 4a^2b^2 \\
\hline
2a^2 - 4ab + b^2 & 2a^2b^2 - 4ab^3 + b^4 \\
 & 2a^2b^2 - 4ab^3 + b^4 \\
\hline\hline
\end{array}$$

The required root is $a^2 - 2ab + b^2$.

Dem. If the quantity, whose root is to be extracted, is a complete square, we shall have its root by observing the way in which the square is made up. If it consist of only three terms it must be the square of a binomial; for the square of every quantity containing more than two terms, contains, itself, more than three. Let us, then, first of all, observe what parts the square of a binomial consists of.

$$(a \pm b)^2 = a^2 \pm 2ab + b^2.$$

That is, the first and last terms of the square are the squares of the two terms of the root, and the second term of the square is twice the product of the two terms of the root (including their respective signs). This will be the case whether the terms ($a$ and $b$) are single letters or numbers or made up each of several factors. Thus $(4x \pm 3y)^2 \doteq (4x)^2 \pm 2.4x.3y + (3y)^2$, since, in this case, we have $4x$ substituted for $a$ and $3y$ for $b$.

If, then, we have a regular square of three terms, let us, in order to find the root, arrange the terms in their proper order, which is evidently according to the powers of the letters. Then the square root of the first term, from what has been said, will be the first term of the root. Now the second term of the square being twice the product of the first and second terms of the root, it follows that, if we divide the second term of the square by twice the first term of the root just found, we have the second term of the root. Thus, let it be required to find the square root of $16x^2 - 24xy + 9y^2$. We begin with the first term $16x^2$, the square root of which is $4x$, by Pr. I., and *that* we take as the first term of the root. Rejecting, then, the term $16x^2$ from the power, by subtracting it, and retaining (or bringing down) the other terms, we know that the next term, $-24xy$, must be made up of twice the product of $4x$ and the second term of the root, or of the product of $8x$ and that term. Consequently the second term of the root $\doteq (-24xy) \div (+8x) = -3y$. We prove this by multiplying $8x$ by $-3y$, and also multiply $-3y$ by $-3y$, since the third term of the power should be the square of $-3y$. If these two products, then, agree with the two remaining terms of the square, leaving no remainder when subtracted, the work is finished and the given quantity is proved to be a complete square.

$$\begin{array}{rl} & 16x^2 - 24xy + 9y^2(4x - 3y \\ & 16x^2 \\ \hline 8x - 3y) & -24xy + 9y^2 \\ & -24xy + 9y^2 \\ \hline \end{array}$$

Next, if the given square consists of more than three terms, the root, consequently, consisting of more than two, we continue the operation in the same way, taking the first two terms of the root as one quantity, and using that to find the third, precisely as we used the first to find the second. Thus, in the Example in the last page, having found the two first terms, $a^2 - 2ab$, in the manner previously described, we take the whole compound quantity $(a^2 - 2ab)$, and regarding it now as one term, we obtain another term, $+b^2$, in the same manner in which we obtained the second term. As a first step to this the square of $(a^2 - 2ab)$ has already been subtracted from the given power in two parts, first $a^4$, and then $-4a^3b + 4a^2b^2$, after which we have merely to divide the remainder by $2(a^2-2ab)$, or $(2a^2-4ab)$, to find the third term $+b^2$.

If the given power consists of more than five terms, we may proceed in the same manner to find the other term or terms of the root.

If the given quantity is not an exact square, we may, by the same method, take the nearest root we can obtain, annexing + or − to it according as a positive or a negative quantity remains when the square of the root found has been subtracted. Or, if desirable, the process may be continued till the root acquire the form of a regular series: but that is seldom, if ever, of any use, unless the first term of the root is greater than the second and subsequent terms, causing the series to *converge*; that is—the terms to become continually less and less, so that, by going on, we have it in our power to arrive at a term less than any assignable quantity, or rather so far that all the subsequent terms together shall be less than any assignable quantity.

NOTE 2. The process for extracting the square roots of numbers is similar to the algebraical operation in form, and the same in principle. Thus, if we are required to extract the square root of 3844, that number being the same as $3600+240+4$, we proceed thus:—

$$\begin{array}{r|l} 60 & 3600+240+4\ (\ 60+2. \\ 60 & 3600 \\ \hline 120+2 & +240+4 \\ & +240+4 \\ \hline\hline \end{array}$$

This adapted to the forms of Arithmetic will be expressed as follows:—

$$\begin{array}{r|l} 6 & 38{,}44\,(\,62. \\ 6 & 36 \\ \hline 122 & 244 \\ & 244 \\ \hline\hline \end{array}$$

EXERCISE 1. Find the square root of $x^2+2xy+y^2$.
Ans. $x+y$.

2. Of $9m^2-12mn+4n^2$. Ans. $3m-2n$.

A. Compute the square root of $16b^2-24bc+9c^2$.

B. Of $64a^2x^2-192a^2xy+144a^2y^2$.

3. Of $x^4-2x^2y^2+y^4+2x^2z^2-2y^2z^2+z^4$.
Ans. $x^2-y^2+z^2$.

4. Of $a^6-4a^5-2a^4+12a^3+9a^2$. Ans. $a^3-2a^2-3a$.

5. Of $4x^2-12xy+9y^2+16xz-24yz+16z^2$.
Ans. $2x-3y+4z$.

6. Of $9x^6-12x^5+10x^4-28x^3+17x^2-8x+16$.
Ans. $3x^3-2x^2+x-4$.

C. Of $49a^2-126ab+154ac+81b^2-198bc+121c^2$.

D. Of $49a^4-70a^3b+67a^2b^2-30ab^3+9b^4$.

NOTE 3. In each of the roots found, the signs may be changed (provided we change all of them), and the result will be another answer equally correct, since $\sqrt{(a^2)}\doteq\pm a$.

NOTE 4. The extraction of other roots than the second is a complicated process, and seldom required.

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# CHAPTER IX.

## COMMON MEASURES.

### DEFINITIONS.

When one quantity is contained a whole number of times in another without leaving a remainder, the former quantity is said to measure the latter, or to be a *Measure* of it. Thus 3 is a measure of 12, and $a+b$ measures $ac+bc$.

When one quantity measures two or more quantities, it is called their *Common Measure.* Thus 3 is a common measure of 9 and 12, 2 is a common measure of all even numbers, and $a^2$ is a common measure of $a^3$ and $a^2b$.

A quantity may have several measures, and two or more quantities may have several common measures.

EXERCISE A. What are the measures of 12?

B. What are the measures of 30?

C. What are the common measures of 12 and 30?

D. What are the measures of $4ab$?

E. What are the measures of $6ac$?

F. What are the common measures of $4ab$ and $6ac$?

### PROBLEM I.

*To find the greatest common Measure of two simple Quantities.*

RULE. The product of all the simple factors common to *both will be the* greatest common measure. *Of the co-effi-*

*cients*, that product may be found, when not evident at first sight, by the usual rule of Arithmetic. *Of the letters*, we have merely to take those common to both quantities. If there are amongst them different powers of the same letter, the lower power of the two must, of course, be taken.

Thus the g. c. m.* of $4ab^4c^3$ and $6abc^5$ is $2abc^3$.

DEM. Let $A$ be the product of all the factors common to both quantities, and let $B$ and $C$ be the products of the remaining factors in each quantity. Then the two quantities are $A.B$ and $A.C$. Now $A$ evidently measures both, and no quantity greater than $A$ can measure both; for, if any number measure $A.B$, it must either be one of the factors of $A$ or of $B$, or the product of two or more of them; and, in like manner, any number that measures $A.C$ must either be one of the factors of $A$ or of $C$, or the product of two or more of them. Every common measure, therefore, must be made up of factors common to both $A.B$ and $A.C$. But there are none such except $A$ or its factors. Therefore $A$ is the g. c. m. of $A.B$ and $A.C$.

We here assume that a number is the product of all its prime factors, or, in other words, that if all the prime numbers which severally measure another number be multiplied together, they will just make that number; and, consequently, that a number cannot be the product of two different sets of prime factors. This is seldom proved in elementary treatises, being regarded as almost self-evident, or much oftener overlooked altogether. Its proof, however, is necessary, to complete the demonstration, but, being too abstruse for beginners, it is reserved for the Key, in which it may be found.

EXERCISE A. What is the g. c. m. of $abc$ and $acd$?
B. What of the quantities $a^3x^2y$ and $a^2bx$?
C. Of $6am^2n$ and $9a^2mn$?
D. Of $238ab^5c^2d$ and $182a^2b^2c^2d^2$?
E. Of $8925mx^4y^3$ and $12852nx^3y^4$?
F. Of $20952ab^3x^5$ and $12528a^5b^2x^2$?

---

* We shall use these letters as an abbreviation for the "greatest common measure"

## PROBLEM II.

### *To find the greatest common Measure of two Quantities, the one simple, the other compound.*

RULE. The product of all the simple factors common to both is the greatest common measure, as in the case of two simple quantities.

The *Demonstration* is the same as that of Problem I.

NOTE. No letter or number is a factor of a compound quantity unless it measures *every term* of that quantity. The common factors among the letters may be found as in Problem I.; and, among the co-efficients, it will generally be seen easily by a little examination; but when it is not, we must begin with the co-efficient of the simple quantity, and the first term of the compound quantity, and, having found their g. c. m., find next the g. c. m. of that and the next term; and so on to the last.

Thus the g. c. m. of $45x^2y^3z^4$ and $315x^4y^2z^3-125x^3y^4z^2$ is $5x^2y^2z^2$.

EXERCISE A. Find the g. c. m. of $42ab^2c^3$ and $14ab-21bc$.

B. Of $110ax^4y^4z$ and $22ax^2y^3-55a^2yz-33axz^2$.

C. Of $546a^3b^4c^5$ and $78a^2bc+273ab^2c-182abc^2$.

## PROBLEM III.

### *To find the greatest common Measure of two compound Quantities.*

CASE 1. *When one of the quantities, or both, consist solely of compound factors.*

RULE. Arrange the terms of each quantity according to the powers of some one letter. Of the two quantities, take that which commences with the higher power (or, if equal, either), and divide it by the other. If there is a remainder, use it as a divisor for the next operation, and by it divide the preceding divisor now used as a dividend. Continue this process until there is no remainder. The last divisor will be the greatest common measure.

DEM. The demonstration of this rule requires to be given at some length. We commence with two almost *self-evident theorems.*

*Theorem* 1.—*If a quantity measure two others, it will measure their sum and also their difference.* Thus, if 5 is contained in 30 six times, and in 20 four times, it is contained in 50 ten times,—that is $(6+4)$ times, because $5 \times (6+4) \doteq (5 \times 6) + (5 \times 4)$; and since 6 and 4 are both whole numbers, $6+4$, or 10, must also be a whole number: therefore 5, being contained in 50 an integral number of times, measures 50. In like manner 5 measures $30-20$, or 10, being contained in it $(6-4)$ times, or 2 times.

*Theorem* 2.—*If a number measure another, it will measure any multiple of it.* This arises obviously from the preceding theorem, since multiplication is equivalent to successive additions of the same quantity.

To apply these theorems to the demonstration of the rule, we may take an arithmetical example, the reasoning employed upon which, it will be observed, will hold good in every similar case, and will be equally applicable to quantities algebraically expressed, because there is nothing in the nature of the proof peculiar to the particular numbers employed for illustration.

Let it be required to find the greatest common measure of 114 and 33, which may properly be taken as instances of compound quantities, since $33 \doteq 30 + 3 = (3 \times 10) + 3$, and $114 \doteq 100 + 10 + 4 = (1 \times 10^2) + (1 \times 10) + 4$. Proceeding by the rule, the work will stand as in the margin, 3 being found for the g. c. m.

```
33)114(3
    99
    —
   15)33(2
      30
      —
      3)15(5
        15
        ==
```

We have, then, two points to make good:—*first*, that 3 must measure both 33 and 114; and *second*, that no number greater than 3 can possibly do so.

To establish the first point, we commence at the end of the operation and trace the steps backward. Since there is no remainder at the close of the process, 3 is contained just 5 times in 15: therefore 3 measures 15; and, if it measure 15, it will measure twice 15, or 30 (Theorem 2). It also measures itself; and if it measure 30 and 3, it must (by Th. 1) measure $30+3$, or 33. If it measure 33 it will measure 3 times 33, or 99. Measuring 99 and 15 it must measure $99+15$, or 114. It therefore measures both 33 and 114, and *must* do so.

But, *secondly*, can no number greater than 3 measure 33 and 114? If it can, it must measure 3 times 33 or 99 (Th. 2), and next 114 − 99, or 15 (Th. 1): if it measure 15 it will measure 30; and if it measure both 33 and 30 it will measure 33 − 30, or 3: but no number greater than 3 can measure 3. Therefore no number greater than 3 can measure both 33 and 114.

Consequently 3 is the g. c. m. of 33 and 114.

Example 1. Find the greatest common measure of the two quantities $x^3+2x^2+2x+1$ and $x^2+2x+1$.

$$\begin{array}{llll}
x^2+2x+1) & x^3+2x^2+2x+1(x & & \\
 & \underline{x^3+2x^2+\ \ x\qquad\qquad\qquad} & & \\
 & & x+1) & x^2+2x+1(x+1 \\
 & & & \underline{x^2+\ \ x\qquad} \\
 & & & \qquad\quad x+1 \\
 & & & \qquad\quad \underline{\underline{x+1}}
\end{array}$$

Ans. $x+1$.

Exercise 1. Find the greatest common measure of $a^3+a^2b-ab^2-b^3$ and $a^2-b^2$. Ans. $a^2-b^2$.

2. What is the greatest common measure of $5-30a+16a^3-6a^4+3a^6$ and $a^5-2a^3+5a^2-10$? Ans. $a^3+5$.

3. Of $x^4-3x^2+2$ and $x^6-3x^4+3x^2-2$? Ans. $x^2-2$.

A. Of $729a^6-243a^4+27a^2-2$, and $81a^4-27a^2+2$?

B. Of $2y^4+5y^3-7y^2-12y+7$, and $2y^3+y^2-11y+5$?

Note 1. When the sign of the first term of any divisor is negative, we may change the signs of *all* the terms if we please.

Note 2. If, in the course of the process just described, the first term of any of the divisors does not measure the first term of the corresponding dividend, divide the former by any simple factor or factors which will divide all its terms, if any such can be found. But if none—or if, after doing so, the first term of the divisor does not yet measure the first term of the dividend, multiply the dividend by the smallest simple factor necessary to effect that object.

Dem. The reason of the permission given in Note 1 is, that if $+a$ be a measure of any quantity $b$, then $-a$ must also measure $b$; for if $b \div (+a) \doteq c$, then $b \div (-a) \doteq -c$.

The direction given in Note 2 rests on the principle that, if $abc$ and $ade$ be two quantities having $a$ for their g. c. m., and consequently the two quantities $bc$ and $de$ incommensurable (that is, without any common factor), then the *g. c. m. will not be changed* if we remove, from either quantity, any of the factors which have no common measure. Thus $a$, being the g. c. m. of $abc$ and $ade$, is also the g. c. m. of $ab$ and $ade$, or of $abc$ and $ad$, since the g. c. m. of two quantities is the product of all the factors common to both. In like manner the g. c. m. of two quantities will not be altered if we introduce a new factor into either, provided that factor has no c. m. with any of the factors of the other quantity. Thus the g. c. m. of $abc$ and $adef$ will also be a, if $f$ contain no factor found in $b$ or $c$.

Now since, in this Case, our given quantities consist of compound factors only, no simple factor can measure either of them. We may therefore, without altering the g. c. m., multiply either of them by any simple factor, or divide any remainder, in which a simple factor makes its appearance, by that factor, or multiply any dividend by any simple factor which is not also a factor of the corresponding divisor, or commensurable with any simple factor contained in it.

**Example** 2. Determine the greatest common measure of $a^3-ab^2$ and $a^2+2ab+b^2$.

$$\begin{array}{r}
a^2+2ab+b^2)a^3 \qquad\quad - ab^2(a. \\
a^3+2a^2b+ ab^2 \\
\hline
-2a^2b-2ab^2.
\end{array}$$

Dividing the remainder (which is to be our next divisor, by $2ab$, according to Note 2, and then changing the signs of both its terms, by Note 1, it becomes $a+b$.

$$\begin{array}{r}
a+b)a^2+2ab+b^2(a+b \\
a^2+\ ab \qquad\quad \\
\hline
ab+b^2 \qquad\quad \\
ab+b^2 \qquad\quad \\
\hline\hline
\end{array}$$

Ans. $a+b$.

**Example** 3. What is the greatest common measure of $3a^2+ab-4b^2$ and $4a^2-5ab+b^2$?

Making the second quantity the dividend, we must multiply it by 3 before its first term can be measured by the first term of the divisor.

$$\begin{array}{r}
4a^2-\ 5ab+\quad b^2 \quad \\
3 \quad \\
\hline
3a^2+ab-4b^2)12a^2-15ab+\ 3b^2(4. \\
12a^2+\ 4ab-16b^2 \quad \\
\hline
-19ab+19b^2 \quad
\end{array}$$

Divide the remainder by $19b$, and change its signs (making $a-b$) for the next divisor.

$$
\begin{array}{l}
a-b)3a^2+\ ab-4b^2(3a+4b. \\
\phantom{a-b)}3a^2-3ab \\
\hline
\phantom{a-b)3a^2+\ }4ab-4b^2 \\
\phantom{a-b)3a^2+\ }4ab-4b^2 \\
\hline\hline
\end{array}
$$

Ans. $a-b$.

Exercise 4. Compute the greatest common measure of $6a^2+7ax-3x^2$ and $6a^2+11ax+3x^2$. Ans. $2a+3x$.

5. The same of $12x^3+4x^2-3x-1$, and $8x^3-4x^2-2x+1$. Ans. $4x^2-1$.

6. Of $x^2+3x-4$ and $x^3-3x^2-10x+36$. Ans. 1.

7. Of $4x^3-8ax^2-a^2x+2a^3$, and $4a^2-x^2$. Ans. $2a-x$.

8. Of $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ and $a^3+x^3-ax^2-a^2x$. Ans. $a^2-2ax+x^2$.

C. Of $8a^4-6a^2+1$, and $64a^6-48a^4+12a^2-2$.

D. Of $125x^3+64y^3$ and $25x^2-16y^2$.

E. Of $8a^2-22ab-21b^2$ and $8a^2-34ab+21b^2$.

Case 2. *When the quantities consist of both simple and compound factors.* Find the greatest common measure of the simple factors, by Problem 1; then that of the compound factors, by Case 1. The product of the two common measures thus found will be the greatest common measure of the two given quantities.

Dem. The direction given for this case follows from what has before been proved—that the g. c. m. of two quantities is the product of all the factors common to both.

Exercise 9. Find the greatest common measure of the two quantities $8(a^2+2ab+b^2)$ and $12(a^3-a^2b-ab^2+b^3)$. Ans. $4(a+b)$.

10. The same of $35x^4-35y^4$ and $63x^2-63y^2$. Ans. $7x^2-7y^2$.

F. Of $12a^5+32a^4b-12a^3b^2$, and $8a^4-72a^2b^2$.

G. Of $6y^4-10y^3+32y$, and $10y^4-10y^3-20y^2+80y$.

Note 3. In some instances the preceding directions and all others are insufficient. In such cases, the greatest common measure can be found only by resolving the two quantities into their factors, which is sometimes a troublesome process.

# CHAPTER X.

## COMMON MULTIPLES.

### DEFINITIONS.

A *Multiple* is the reverse of a measure. When one quantity measures another, the latter is said to be a multiple of the former. Thus 20 is a multiple of 5, and $ab$ is a multiple of $a$.

When one quantity is a multiple of two or more quantities, it is called their *Common Multiple*. Thus 12 is a common multiple of 1, 2, 3, 4, and 6. Thus, also, $abc$ is a common multiple of $a$, $b$, $ab$, $ac$, and $bc$.

### PROBLEM.

*To find the least common Multiple of two or more given Quantities.*

RULE. Strike out any of the quantities, or of their coefficients or literal factors, which may be found to measure any of the others, but not such as measure quantities or factors which are themselves struck out.

Write the remaining quantities in a line by themselves, and divide as many of them as possible by any prime number or quantity that will measure two or more.

Write the several quotients and undivided quantities in a second line, and in that line again cancel any quantity or co-efficient that measures any other.

Proceed in the same way with the remaining quantities, continuing the process until it appears that no two of the quantities left have any common measure. Then the continued product of the several divisors, and of the quantities in the last line, will be the least common multiple required.

DEM. The least common multiple of any number of quantities is evidently that expression which contains every factor found in any one of the quantities, but without a repetition of those found in more than one. We secure both points by omitting, in the first place, all such factors or quantities as are contained in other quantities, and, afterwards, when we divide several numbers by one factor, by again taking in that factor once.

Note. It will be seen by inspection when any simple literal quantity measures any other quantity. In the case of numbers and compound quantities it will also, in most instances, be readily perceived, after a little experience. But when it is not so it may be ascertained by trial.

The same experience will also often indicate the proper divisors; but when a divisor is not readily perceived, it may be found for the co-efficients by trying the successive prime numbers. Both with the co-efficients and compound quantities we shall very rarely err,* and never make any error of practical importance, if we use for a divisor the greatest common measure of any two of the co-efficients or compound quantities.

Example 1. What is the least common multiple of the quantities $ax^2$, $x$, and $a^2y$.

Here we strike out $x$ because it measures $ax^2$, and $a$ because it measures $a^2$. Then, since there are no co-efficients or compound quantities, the operation is finished by taking the product of the remaining quantities, viz. of $x^2$ and $a^2y$, which is $a^2x^2y$.

Exercise 1. Find the least common multiple of $ab$ and $ac$. Ans. $abc$.

2. Of $xy$, $yz$, and $xz$. Ans. $xyz$.

A. Of $a$, $b$, $ax$, $by$, and $xy$.

B. Of $a^2b^3c$, and $ab^2c^3$.

C. Of $x^3y^2z^2$, $x^2y^2z^3$, and $x^2y^3z^2$.

Example 2. What is the least common multiple of 15, 54, 48, and 60?

15 is omitted because it measures 60.

$$\begin{array}{r} 2\underline{)54,\ 48,\ 60} \\ 3\underline{)27,\ 24,\ 30} \\ 2\underline{)9,\ \ 8,\ 10,} \\ 9,\ \ 4,\ \ 5. \end{array}$$

$$2\times3\times2\times9\times4\times5 \doteq 2160, \text{ Ans.}$$

Exercise 3. What is the least common multiple of 12, 15, and 20? Ans. 60.

D. Of 144 and 96?

E. Of 36, 80, 72, and 100?

---

* The case of possible error will be pointed out in the Key.

**Example** 3. Compute the least common multiple of the quantities $2a$, $15a^2$, $3b$, $6ab$, $4c$, $12ac$, and $50bc$.

We first cancel $2a$, then $3b$, because each of them measures $6ab$. We next reject $4c$ since it measures $12ac$; and after that the factors $a$, $b$, $a$ again, and $c$, because they severally measure other factors. Our quantities then stand as follows:—

$$\begin{array}{l} 5\underline{)15a^2,\ 12,\ 50bc} \\ 2\underline{)3a^2,\ 12,\ 10bc} \\ \quad a^2,\ \ 6,\ \ 5bc. \end{array}$$

In the second line 3 is expunged because it measures 12. The least c. mult. is therefore $5 \times 2 \times a^2 \times 6 \times 5bc = 300a^2bc$.

The cancelled factors are merely omitted in the process as just exhibited, but in practice it is better to draw the pen through them.

**Exercise** 4. Find the least common multiple of $8a^3b^2$ and $20a^2b^3$. Ans. $40a^3b^3$.

5. Of $3a$, $4b$, $5c$, $8a^2b$, $12ac^2$, and $15b^2c$. Ans. $120a^2b^2c^2$.

6. Of $7x$, $9y$, and $11z$. Ans. $693xyz$.

7. Of $14x^2$, $21xy$, $6yz$, $20xz$, and $15z^2$. Ans. $420x^2yz^2$.

F. Of $75ab$ and $100ac$.

G. Of $48x^2y$ and $60xy^2$.

H. Of $14p$, $21q$, and $24r$.

K. Of $48ab$, $60bc$, $35cd$, and $42de$.

L. Of $8a$, $24a$, $18a$, $54a$, $9a$, $15a$, and $27a$.

M. Of $5a$, $6b$, $7c$, and $11d$.

**Example** 4. What is the least common multiple of the quantities $a+b$, $a^2-b^2$, $a^2+ab-2b^2$, and $a^2-ab-6b^2$?

We first omit $a+b$, because it measures $a^2-b^2$.

We next find the g. c. m. of $a^2-b^2$ and $a^2+ab-2b^2$ to be $a-b$, which we use as our first divisor, and find that it will not divide any other of the given quantities than these two. The process therefore stands thus,—

$$\begin{array}{l} a-b\underline{)a^2-b^2,\ a^2+ab-2b^2,\ a^2-ab-6b^2} \\ \qquad a\ +b,\qquad a+2b,\qquad a^2-ab-6b^2. \end{array}$$

We now perceive at once that $a+b$ and $a+2b$ have no common measure: we therefore try if $a+b$ will divide $a^2-ab-6b^2$, and find it will not. Our next trial is with $a+2b$, which, we find measures $a^2-ab-6b^2$, and is therefore to be struck out.

Consequently the least common multiple required is $(a-b) \times (a+b) \times (a^2-ab-6b^2) = a^4 - a^3b - 7a^2b^2 + ab^3 + 6b^4$.

Exercise 8. What is the least common multiple of $4x^2 + 12x$ and $5x + 15$? Ans. $20x^2 + 60x$.

9. Of $15a^2$, $5a + 10x$, and $ax + 2x^2$? Ans. $15a^3x + 30a^2x^2$.

10. Of $4x^2 - 4xy + y^2$, and $4x^2 - y^2$? Ans. $8x^3 - 4x^2y - 2xy^2 + y^3$.

N. Of $3a^2 + 12ab$, and $5ab + 20b^2$?

O. Of $a + 6$ and $a^2 + 12a + 36$?

P. Of $25 - 10x + x^2$ and $50 - 2x^2$?

11. Of $x^2 - y^2$, $x^2 + 2xy + y^2$, and $x^2 - 2xy + y^2$? Ans. $x^4 - 2x^2y^2 + y^4$.

12. Of $x - 2$, $x + 2$, $x^2 + 4$, and $x^2 - 4$? Ans. $x^4 - 16$.

13. Of $z^3 + z^2 + z + 1$ and $z^3 - z^2 + z - 1$? Ans. $z^4 - 1$.

14. Of $2m - 5$ and $2m + 7$. Ans. $4m^2 + 4m - 35$.

Q. Of $9a^2 - 16b^2$, $9a^2 + 24ab + 16b^2$, and $9a^2 - 24ab + 16b^2$?

R. Of $8b^3 - 12b^2c + 18bc^2 - 27c^3$ and $8b^3 + 12b^2c + 18bc^2 + 27c^3$?

## CHAPTER XI.

### FRACTIONS.

Note 1. The *Definitions* of the terms usually employed in fractions are the same as in common Arithmetic. It is therefore needless to repeat them here.

Note 2. There are two ways in which a fraction may be defined. These are sometimes both employed by the same writer, and confounded with each other as if they were identical, whereas they are essentially distinct. Thus 2 thirds may either be defined to be 2 divided by 3, or 2 of the 3 equal parts into which a unit is divided. These two definitions are, no doubt, consistent; but they must be proved to be so and not assumed. In the following demonstrations the latter is taken as the true definition.

### Problem I.

*To reduce an integral Quantity to a Fraction having a given Denominator.*

*Rule.* Multiply the whole number by the given denominator, for the numerator of the fraction, and write under it the given denominator.

**Dem.** If a unit is divided into $n$ equal parts, then 1 unit contains $n$ such parts, and $m$ units contain $mn$ such parts. That is, $1 \doteq \frac{n}{n}$, and $m \doteq \frac{mn}{n}$. If therefore $m$ and $n$ are given the numerator of the required fraction will be $mn$.

**Exercise** 1. Reduce 7 to a fraction having its denominator 4, and $a$ to a fraction having its denominator $b$.

Answers: $\frac{28}{4}$ and $\frac{ab}{b}$.

2. Change $7xy^2$ into a fraction whose denominator shall be $5xyz$, and $a+x$ into a fraction having its denominator $a-x$.

Answers: $\frac{35x^2y^3z}{5xyz}$ and $\frac{a^2-x^2}{a-x}$.

A. Reduce $15z$ to a fraction having its den. 36.
B. Reduce $a-x$ to a fraction having its den. $a+x$.
C. Change $5ax$ into a fraction having 1 for its den.
D. Convert the integral quantity $y^2+7y+49$ into a fraction whose den. is given, viz. $y-7$.

## Problem II.

### *To reduce a mixed Quantity to the Form of a Fraction.*

**Rule.** Multiply the whole number by the denominator of the given fraction, and add its numerator to the product, if the sign of the fraction is positive; but subtract it, if negative. Write the result for the numerator of the required fraction, and, for its denominator, that of the given fraction.

**Dem.** Since $m \doteq \frac{mn}{n}$, $m+\frac{p}{n} \doteq \frac{mn}{n}+\frac{p}{n}=\frac{mn+p}{n}$, for $mn$ parts added to $p$ parts must make $mn+p$ parts of the same kind, the denominator $n$ merely indicating what kind of parts they are.

In like manner $m-\frac{p}{n} \doteq \frac{mn}{n}-\frac{p}{n}=\frac{mn-p}{n}$.

**Exercises.** Reduce the following mixed numbers to fractional forms:

1. $a+\frac{b^2}{c}$. Ans. $\frac{ac+b^2}{c}$.

2. $x-\frac{y^2}{2z}$. Ans. $\frac{2xz-y^2}{2z}$.

3. $3a-\frac{4ax+a^2}{4x}$. Ans. $\frac{8ax-a^2}{4x}$.

4. $a^2-b^2+\frac{b^4}{a^2+b^2}$. Ans. $\frac{a^4}{a^2+b^2}$.

5. $3x-4y-\frac{5xy+12y^2}{4x-3y}$. Ans. $\frac{12x^2-30xy}{4x-3y}$.

6. $8n-3-\frac{8mn-6m}{m-1}$. Ans. $\frac{3m-8n+3}{m-1}$.

7. $x-y-\frac{x^3+y^3}{x^2+xy+y^2}$. Ans. $-\frac{2y^3}{x^2+xy+y^2}$.

A. $3a-\frac{7x^2}{5z}$.

B. $7x^2+\frac{3y^5}{5x^3}$.

C. $7a-14b+\frac{5a^2+3b^2}{8a}$.

D. $8x+\frac{11x^2-16xy+11y^2}{6y}$.

E. $7y-9z-\frac{49y^2+81z^2}{7y+9z}$.

F. $x^2-9-\frac{x^6-729}{x^4+9x^2+81}$.

G. $a^4+ax^3+\frac{ax^6-x^7}{a^3-x^3}$.

## PROBLEM III.

*To reduce an improper Fraction to the Form of an integral, or of a mixed, Quantity.*

**RULE.** Divide the numerator by the denominator.

**DEM.** The fraction $\frac{m}{n}$, as has been said before, means that a unit is divided into $n$ equal parts, and that $m$ such parts constitute the fraction. This definition is alike applicable whether $m$ be less than $n$, greater than $n$, or equal to it. We have now to prove that the fraction $\frac{m}{n} = m \div n$. *The quotient* $m \div n$ means the $n$th part of $m$: but evidently we shall have the $n$th part of $m$ if we have the $n$th part of all the separate units into which $m$ is divided. But, by our definition of a fraction, the $n$th part of one of those units is $\frac{1}{n}$, and in $m$ units we have $m$ such parts, or $\frac{m}{n}$. $\therefore \frac{m}{n} = m \div n$. Consequently, when $m$ is greater than $n$, if we actually divide $m$ by $n$, we shall have the same quantity that the fraction expresses, only in a different form, viz. in that form required by the problem.

Or, since the object of this problem is the reverse of that of the two preceding problems, that object will be attained by reversing their process: in the two previous problems we multiply by the given denominator; in this we divide by it.

Exercises. Change the following fractions into integral or mixed quantities.

1. $\frac{ac+b^2}{c}$. Ans. $a+\frac{b^2}{c}$.

2. $\frac{2xz-y^2}{2z}$. Ans. $x-\frac{y^2}{2z}$.

3. $\frac{2x^3-2y^3}{x^2+xy+y^2}$. Ans. $2x-2y$.

4. $\frac{a^4}{a^2+b^2}$. Ans. $a^2-b^2+\frac{b^4}{a^2+b^2}$.

5. $\frac{12x^2-30xy}{4x-3y}$ Ans. $3x-\frac{21xy}{4x-3y}$.

6. $\frac{3m-8n+3}{m-1}$. Ans. $3-\frac{8n-6}{m-1}$.

A. $\frac{35x^5+3y^5}{5x^3}$, and $\frac{8ax-a^2}{4x}$, separately.

B. $\frac{45x^5y^2-36x^2y^5}{9x^2y^2}$.

C. $\frac{12a^2-20b^2}{3a-6b}$, and $\frac{8a^3+27b^3}{2a+3b}$.

D. $\frac{48y^2-46yz}{6y-11z}$, and $\frac{24x^3-13x^2}{8x^2+9x-10}$.

E. $\frac{a^2-4b^2}{a+2b-3c}$, and $\frac{a^7-x^7}{a^3-x^3}$.

### THEOREM.

*A Fraction is not changed in Value by multiplying or dividing both terms by the same quantity.*

DEM. When the denominator is doubled, it implies that the unit is divided into twice as many parts, and, consequently, that each part is only half the size of one of the original parts. It therefore requires twice as many such parts to make up the same quantity, and twice as many such parts are obtained by doubling the numerator. In general, if we have the fraction $\frac{a}{b}$ and multiply both terms by $m$, making $\frac{ma}{mb}$, we have $m$ times the previous number of parts, and each part $m$ times smaller, or rather the $m$th part of one of the original parts. Consequently the fraction is equivalent to the original fraction, or $\frac{ma}{mb} \doteq \frac{a}{b}$. Now, if the former of these fractions is obtained from the latter by multiplying both terms by $m$, the latter may be obtained from the former by dividing both terms by $m$. Therefore both cases of the theorem are established.

EXERCISE A. Whether is $\frac{3}{4}$ or $\frac{15}{20}$ the greater?

B. Is $\frac{4a}{5a}$ greater or less than $\frac{4}{5}$?

DEFINITION. Any quantity by which both terms are multiplied or divided is called their *common Multiplier or common Divisor*. The former is placed to the right of the fraction, as $\frac{3}{4}(2$; the latter to the left, as $2)\frac{6}{8}$. Thus $\frac{3}{4}(2 = \frac{6}{8}$, and $2)\frac{6}{8} = \frac{3}{4}$.

### PROBLEM IV.

*To change one Fraction into another of equal Value having a given greater Denominator.*[*]

RULE. Divide the greater denominator by the smaller for the common multiplier. Place that to the right of the given fraction, and multiply the numerator by it.

---

[*] It is not absolutely necessary that the greater denominator be a multiple of the smaller. When not so, we are merely driven into the form of a complex *fraction.*

The *Demonstration* is contained in the preceding theorem.

**Example.** Change $\frac{a}{2b}$ into a fraction having its denominator $6bc$. $\frac{a}{2b}(3c=\frac{3ac}{6bc}$. Ans.

**Exercise** 1. Convert $\frac{7}{a-b}$ into a fraction having its denominator $2a^2-2ab$. Ans. $\frac{14a}{2a^2-2ab}$.

2. Alter the form of the fraction $\frac{5}{a+x}$, so that its denominator may be $a^2-x^2$. Ans. $\frac{5a-5x}{a^2-x^2}$.

3. What fraction, with the denominator $4a^2-12a+9$, is equivalent to the fraction $\frac{2a+3}{2a-3}$? Ans. $\frac{4a^2-9}{4a^2-12a+9}$.

A. Reduce $\frac{x}{y}$ to a fraction having its denominator $y^2$.

B. Change $\frac{2a}{3}$ into a fraction having its den. $12ab$.

C. Reduce $\frac{3m}{4n}$ to the denominator $20m^2n^2$.

D. Change $\frac{3y^2-5z^2}{a}$ to the denominator $2a^2b^2$.

E. Change $\frac{7a^2-4b^2}{x}$ into a fraction having its denominator $2ax+3bx$.

F. Convert $\frac{a+x}{a-x}$ into an equivalent fraction having its denominator $(a-x)^2$.

## Problem V.

### *To reduce a Fraction to its lowest Terms.*

**Rule.** Cancel all the factors common to the numerator and denominator; or, in other words, divide both by their greatest common measure.

**Dem.** A fraction is in its lowest terms when its terms have no common measure greater than 1. If, then, the terms of a fraction are both divided by their g. c. m., that is, by all the factors common to both, there remains no other factor common to both, by which they can afterwards be divided; or, in other words, the fraction has been reduced to its lowest terms.

EXAMPLE. Reduce $\frac{a^3+b^3}{a^2-b^2}$ to its lowest terms.

We first find the g. c. m. to be $a+b$. Then

$$a+b)\frac{a^3+b^3}{a^2-b^2} \doteq \frac{a^2-ab+b^2}{a-b}.$$

NOTE 1. It is immaterial whether we divide by the g. c. m. at once, or by different common measures in succession.

EXERCISES. Reduce the following to their lowest terms:

1. $\frac{xy}{y^2}$, and $\frac{15m^3n}{20m^2n^2}$. Answers: $\frac{x}{y}$, and $\frac{3m}{4n}$.

2. $\frac{959a^5b^3c}{685a^2b^4c^3}$, and $\frac{5a-5x}{a^2-x^2}$. Answers: $\frac{7a^3}{5bc^2}$, and $\frac{5}{a+x}$.

3. $\frac{m^2+3m+2}{m^3+3m^2+4m+2}$. Ans. $\frac{m+2}{m^2+2m+2}$.

4. $\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$. Ans. $\frac{a^2+x^2}{a-x}$.

A. $\frac{3ac}{6bc}$, and $\frac{888x^4y^3z^2}{925x^2y^3z^4}$.

B. $\frac{x^2+2xy+y^2}{x^2-y^2}$, and $\frac{x^2-y^2}{x^2-2xy+y^2}$.

C. $\frac{a^2+2ab+b^2}{a^3+b^3}$, and $\frac{a^4-b^4}{a^3-b^3}$.

D. $\frac{4a^3+a^2b-5ab^2}{4a^2+9ab+5b^2}$, and $\frac{12x^2-17xy-5y^2}{9x^3+21x^2y-60xy^2}$.

NOTE 2. When the terms of a fraction are compound quantities, and any simple factor is found to measure both, the operation is facilitated by first dividing the terms by the simple factor, and then afterwards by any other common measures in succession, or by the greatest common measure of the reduced terms at once.

EXAMPLE. Reduce $\frac{7a^3-28ab^2}{7a^3+35a^2b+42ab^2}$ to its lowest terms.

$$7a)\frac{7a^3-28ab^2}{7a^3+35a^2b+42ab^3} = \frac{a^2-4b^2}{a^2+5ab+6b^2}$$

$$a+2b)\frac{a^2-4b^2}{a^2+5ab+6b^2} = \frac{a-2b}{a+3b}.$$

**Exercise** 5. Reduce $\frac{a^5 - 2a^4 - 15a^3}{a^4 - 8a^3 + 15a^2}$ to its lowest terms.

Ans. $\frac{a^2 + 3a}{a - 3}$.

6. The same with $\frac{20(a^2 - 2ab + b^2)}{35(a^3 - 2a^2b + 2ab^2 - b^3)}$.

Ans. $\frac{4(a - b)}{7(a^2 - ab + b^2)}$.

E. The same with $\frac{114(a + b)^3}{152(a + b)^2}$.

F. The same with $\frac{10y^3 + 22y^2 - 24y}{10y^3 - 38y^2 + 24y}$.

## Problem VI.

*To reduce Fractions having different Denominators to equivalent Fractions having the same Denominator.*

**Rule.** Take any common multiple of all the denominators for the common denominator, and reduce all the fractions to that, by Problem IV.

**Note.** The common multiple most easily found is the product of all the denominators; and, in that case, every numerator is multiplied by all the denominators but its own, for the new numerators. But much work is saved, in many operations, by using the *least* common multiple, found by the rule in the last Chapter.

The rule needs no *Demonstration* further than that given in the theorem preceding Problem IV.

**Example.** Reduce $\frac{a}{4xy}$, $\frac{b}{6y^2}$, and $\frac{c}{10yz}$, to fractions having a common denominator.

We first find the least common multiple, by Ch. X., to be $60xy^2z$, and then proceed thus:—

$$\frac{a}{4xy}(15yz = \frac{15ayz}{60xy^2z}.$$

$$\frac{b}{6y^2}(10xz = \frac{10bxz}{60xy^2z}.$$

$$\frac{c}{10yz}(6xy = \frac{6cxy}{60xy^2z}.$$

**Exercise** 1. Reduce $\frac{a}{x}$, $\frac{b}{y}$, and $\frac{c}{z}$ to the same denominator.

Ans. $\frac{ayz}{xyz}$, $\frac{bxz}{xyz}$, and $\frac{cxy}{xyz}$.

2. Reduce $\frac{a+x}{a-x}$ and $\frac{a-x}{a+x}$ to fractions having a common denominator.

Ans. $\frac{a^2+2ax+x^2}{a^2-x^2}$ and $\frac{a^2-2ax+x^2}{a^2-x^2}$.

3. The same with $\frac{1}{a^2+2ax+x^2}$, and $\frac{1}{a^2-x^2}$, and $\frac{1}{a^2-2ax+x^2}$.

Ans. $\frac{a^2-2ax+x^2}{a^4-2a^2x^2+x^4}$, $\frac{a^2-x^2}{a^4-2a^2x^2+x^4}$, and $\frac{a^2+2ax+x^2}{a^4-2a^2x^2+x^4}$.

A. Reduce $\frac{x}{4ab}$ and $\frac{y}{6ac}$ to a common denominator.

B. The same with $\frac{5}{6a}$, $\frac{3}{4b}$, and $\frac{4}{5c}$.

C. The same with $\frac{a}{2}$, $\frac{2b}{3}$, and $\frac{3c}{4}$.

D. The same with $\frac{2a}{15x^2y}$, and $\frac{3a}{20xy^2}$.

E. $\frac{5x}{12m^2}$, $\frac{7x}{20mn}$, and $\frac{8x}{21n^2}$.

F. $\frac{3a+5b}{18x}$, $\frac{8a-10b}{27y}$, and $\frac{9a^2-6b^2}{54xy}$.

G. $\frac{x+6}{x-6}$, and $\frac{x-6}{x+6}$.

H. $\frac{a^2+10a+25}{a^2-10a+25}$, $\frac{a^2+25}{a^2-25}$, and $\frac{a^2-10a+25}{a^2+10a+25}$.

## Problem VII.

### *To add Fractions together.*

Rule. *If they have all the same denominator*, add their numerators, and under their sum, as a numerator, write the common denominator.

But, *if they have not the same denominator*, reduce them to a common denominator, and then proceed according to the foregoing direction.

Dem. It scarcely requires proof that $\frac{3}{7}+\frac{2}{7}=\frac{5}{7}$, for the denominator 7 merely indicates what kind of parts they are, while the numerator counts their number. Since, then, *3 pounds and 2* pounds together make 5 pounds, or 3 feet and 2 feet make 5 feet, in like manner 3 sevenths and 2 sevenths will make 5 sevenths. The same explanation will apply to all other fractions having their denominators the same; and, if they have not, they can first be changed into equivalent fractions having the same denominator, by Problem VI., and then their numerators may be added together.

NOTE. If the result does not come out at once in its lowest terms, it should be reduced to them, and, when possible, to an integral form.

EXAMPLE. What is the sum of the following fractions: $\frac{a}{a+x}$, $\frac{x}{a-x}$, and $\frac{2ax}{a^2-x^2}$?

$$\frac{a}{a+x}\,(a-x = \frac{a^2-ax}{a^2-x^2}$$
$$\frac{x}{a-x}\,(a+x = \frac{ax+x^2}{a^2-x^2}$$
$$\frac{2ax}{a^2-x^2}\,(1 \quad = \frac{2ax}{a^2-x^2}$$
$$a+x)\frac{a^2+2ax+x^2}{a^2-x^2} = \frac{a+x}{a-x}.\ \text{Ans.}$$

EXERCISE 1. Add together $\frac{3}{a}$, $\frac{5}{a}$, and $\frac{7}{a}$. Ans. $\frac{15}{a}$.

2. $\frac{3a+b}{5x}$, $\frac{2a-3b}{5x}$, and $\frac{-a+2b}{5x}$. Ans. $\frac{4a}{5x}$.

3. $\frac{5a}{8}$, $\frac{7a}{8}$, $\frac{9a}{8}$, and $\frac{11a}{8}$. Ans. $4a$.

A. $\frac{5}{xy}$, $\frac{19}{xy}$, $\frac{12}{xy}$, and $\frac{32}{xy}$.

B. $\frac{3a}{53}$, $\frac{9a}{53}$, $\frac{15a}{53}$, and $\frac{16a}{53}$.

C. $\frac{a}{3}$, $\frac{2a}{3}$, $\frac{4a}{3}$, and $\frac{5a}{3}$.

D. $\frac{19}{5x}$, $\frac{20}{5x}$, and $\frac{21}{5x}$.

E. $\frac{x^2+2xy+y^2}{z}$, $\frac{x^2-2xy+y^2}{z}$, and $\frac{x^2-y^2}{z}$.

F. $\frac{a+b}{2}$, and $\frac{a-b}{2}$.

4. $\frac{x}{4}$, $\frac{x}{5}$, $\frac{x}{6}$, and $\frac{x}{10}$. Ans. $\frac{43x}{60}$.

5, $\frac{a}{b}$ and $\frac{c}{d}$. Ans. $\frac{ad+bc}{bd}$.

6. $\frac{7m+12n}{8}$ and $\frac{12m-7n}{10}$. Ans. $\frac{83m+32n}{40}$.

7. $\frac{2a}{3b}$, $\frac{3b}{4c}$, and $\frac{4c}{5a}$. Ans. $\frac{40a^2c+45ab^2+48bc^2}{60abc}$.

8. $\frac{a}{x^2y}$ and $\frac{a}{xy^2}$. Ans. $\frac{ax+ay}{x^2y^2}$.

G. $\frac{3a}{2}$, and $\frac{2a}{3}$.

H. $\frac{3}{4m^2}$, $\frac{4}{5mn}$, and $\frac{5}{6n^2}$.

K. $\frac{c}{2}$, $\frac{2c}{3}$, $\frac{3c}{4}$, and $\frac{4c}{5}$.

L. $\frac{3a}{8bc}$, $\frac{5b}{6ac}$, and $\frac{8c}{9ab}$.

9. $\frac{m+n}{m-n}$ and $\frac{m-n}{m+n}$. Ans. $\frac{2m^2+2n^2}{m^2-n^2}$.

10. $\frac{1}{y^2+2yz+z^2}$, $\frac{1}{y^2-z^2}$, and $\frac{1}{y^2-2yz+z^2}$.

Ans. $\frac{3y^2+z^2}{y^4-2y^2z^2+z^4}$.

M. $\frac{4ax}{a^2-x^2}$, and $\frac{a-x}{a+x}$.

N. $\frac{a}{a+b}$, and $\frac{a}{a-b}$.

O. $\frac{2a^2-3b^2}{6a^2-8ab}$, and $\frac{2a^2+3b^2}{15ab-20b^2}$.

P. $\frac{2a}{a^2+2ax+x^2}$, and $\frac{2x}{a^2-x^2}$.

## PROBLEM VIII.

*To subtract one Fraction from another.*

**RULE.** Proceed as in addition of fractions, subtracting instead of adding.

The *principle* is the same as in addition.

**EXERCISE** 1. From $\frac{38mx}{ay}$ take $\frac{15mx}{ay}$. Ans. $\frac{23mx}{ay}$.

A. From $\frac{98x}{y}$ take $\frac{35x}{y}$.

B. From $\frac{15a}{16}$ subtract $\frac{37a}{16}$.

C. From $\frac{5a^3+7a^2}{13b}$ take $\frac{2a^3-3a^2}{13b}$.

2. From $\frac{10a}{21}$ subtract $\frac{9b}{14}$. Ans. $\frac{20a-27b}{42}$.

D. From $\frac{5m}{6}$ subtract $\frac{3m}{4}$.

E. Collect into one fraction, $\frac{a}{x}-\frac{b}{y}$.

F. $\frac{m}{16a^2b}-\frac{n}{24ab^2}$.

G. $\frac{5}{102yz}-\frac{4}{85yz}$.

3. $\frac{9x-16}{4}-\frac{6x-20}{5}$. Ans. $\frac{21x}{20}$.

H. Reduce to the simplest form, $\frac{a+b}{2}-\frac{a-b}{2}$.

K. $\frac{15a+14b}{20}-\frac{9a+8b}{12}$.

L. $\frac{35a^3-48a^2+20a}{2x}-\frac{30a^3-72a^2-5a}{3x}$.

4. $\frac{a+x}{a-x}-\frac{a-x}{a+x}$. Ans. $\frac{4ax}{a^2-x^2}$.

5. $\frac{1}{2a-2b}-\frac{1}{2a+2b}$. Ans. $\frac{b}{a^2-b^2}$.

6. $\frac{2a}{a^2+2ax+x^2}-\frac{2x}{a^2-x^2}$. Ans. $\frac{2a^2-4ax-2x^2}{a^3+a^2x-ax^2-x^3}$.

M. Subtract $\frac{7}{5a+4b}$ from $\frac{7}{5a-4b}$.

N. Take $\frac{4a^2-9x^2}{(2a+3x)^2}$ from $\frac{4a^2+9x^2}{4a^2-9x^2}$.

NOTE. If we are required to add together, or to collect into one sum, several fractions, some of which have positive signs and others negative, it is the same thing as if we had to add together those with positive signs, and to subtract the others, taking their negative signs merely as indications of subtraction to be performed. We therefore, after reducing all to the same denominator, change the signs of all the terms in the numerators of those fractions which are preceded by the negative sign, and then add all the numerators together.

In doing this we must keep in mind that the sign prefixed to the fraction belongs to the whole fraction, and not to the first term of its numerator; and also that, when the first term of the numerator has no sign of its own prefixed to it, it is understood to be positive, and when changed becomes negative.

EXERCISE 7. What is the sum of $+\frac{c}{15ab}$, $+\frac{a}{21bc}$, and $-\frac{b}{28ac}$? Ans. $\frac{20a^2-15b^2+28c^2}{420abc}$.

8. $\frac{a+2b}{x}$, $-\frac{a+b}{x}$, and $\frac{2a+b}{x}$? Ans. $\frac{2a+2b}{x}$.

9. $\frac{a^2+2ax+x^2}{a-x}$, and $-\frac{a^2-2ax+x^2}{a+x}$. Ans. $\frac{6a^2x+2x^3}{a^2-x^2}$.

0. $+\frac{a+b}{b}$, $-\frac{a}{a+b}$, and $-\frac{a^3-a^2b}{a^2b-b^3}$.

## PROBLEM IX.

### *To Multiply one Fraction by another, or to find the Product of any Number of Fractions.*

RULE. Multiply all the numerators together for the numerator of the product, and all the denominators for the denominator of the product.

*The sign* follows the same rule as in the case of integral quantities.

DEM. When we say £7, and when we say £$\frac{2}{3}$, or 7 pounds and $\frac{2}{3}$ of a pound, we, in both instances, express the number of pounds; but, when that number is fractional, we vary the mode of expression. Thus 7 pounds and $\frac{21}{3}$ of a pound are identical, the 7 and the $\frac{21}{3}$ alike expressing the number of pounds. So 7 fives, and $\frac{2}{3}$ of 5, alike express a certain number of fives. But 7 fives is the same as 7 times 5, or $5 \times 7$: so $\frac{2}{3}$ of 5 is the same as $5 \times \frac{2}{3}$. One idea is attached to both expressions, although the form of expression is different.* In like manner $\frac{2}{3}$ of $\frac{5}{7}$ is the same as $\frac{5}{7} \times \frac{2}{3}$.

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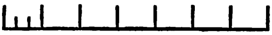
* Wood's Algebra (30).

When, therefore, we are called upon to multiply a fraction by a fraction, the clearest idea of what is intended is obtained by changing the form of expression. When it is said "multiply $\frac{5}{7}$ by $\frac{2}{3}$" it is meant "take $\frac{2}{3}$ of $\frac{5}{7}$." Having then ascertained the true meaning of "multiplying a fraction by a fraction," we may next investigate how the object intended is to be attained. It is evident that we shall readily obtain *two* thirds of $\frac{5}{7}$ if we can obtain *one* third of it; and it is equally evident that we may easily obtain one third of *five* sevenths if we can find one third of *one* seventh.

In order to have 1 seventh we divide the unit into seven equal parts, and take one of them; and 1 third of that one will be obtained by dividing *it* into three equal parts. But if we divide each of the sevenths into three parts, we have 7 threes, or 21 parts altogether, and one of these is the 21st part of the unit. Therefore $\frac{1}{3}$ of $\frac{1}{7} \doteq \frac{1}{21}$. But $\frac{1}{3}$ of $\frac{5}{7}$ is obtained by taking $\frac{1}{3}$ of $\frac{1}{7}$ five times, that is five times $\frac{1}{21}$, or $\frac{5}{21}$. Again, if *one* third of $\frac{5}{7} \doteq \frac{5}{21}$, *two* thirds must be twice as much. But twice 5 makes 10: therefore twice $\frac{5}{21} \doteq \frac{10}{21}$. Consequently $\frac{2}{3}$ of $\frac{5}{7} \doteq \frac{10}{21}$.



Now the 10 was obtained by multiplying 5 by 2; and the 21, by multiplying 3 by 7. Hence the rule.

The process of *cancelling* the common factors, as directed in Note 1, is merely anticipating the reduction of the fraction to its lowest terms. It avoids the unnecessary labour of first multiplying and then dividing.

EXAMPLE 1. Multiply $\frac{2a^2-3b^2}{x}$ by $\frac{2a^2+3b^2}{y}$.

$$\frac{2a^2-3b^2}{x} \times \frac{2a^2+3b^2}{y} \doteq \frac{4a^4-9b^4}{xy}.$$

EXERCISE A. Multiply $\frac{a}{b}$ by $\frac{c}{d}$.

B. Multiply $-\frac{2x}{7y}$ by $-\frac{3x^2}{5y^2}$.

C. Multiply $+\frac{27a^2x^2}{35b^4y}$ by $-\frac{6a^3x}{11by^2}$.

D. What is the product of $\frac{2a+x}{2a-x}$ and $\frac{a+2x}{a-2x}$?

EXAMPLE 2. Multiply together $-\frac{3a^2b}{7x^3}$, $-\frac{5ab^2}{8y^3}$, and $-\frac{9abc}{14xyz}$.

$$\left(-\frac{3a^2b}{7x^3}\right)\times\left(-\frac{5ab^2}{8y^3}\right)\times\left(-\frac{9abc}{14xyz}\right)=-\frac{135a^4b^4c}{784x^4y^4z}.$$

E. What is the product of $-\frac{3a}{x}$, $+\frac{4b}{5y}$, and $-\frac{6c^2}{7xy}$?

F. Of $\frac{x-7}{x-3}$, $\frac{x+7}{x+3}$, and $\frac{x^2+49}{x^2+9}$.

NOTE 1. The same factor or factors occurring in any numerator, and also in a denominator, may be cancelled in both. In practice they are cancelled by drawing the pen through them, a process which cannot properly be exhibited in print.

EXAMPLE 3. Express in its simplest form $\frac{3a}{7b}\times\frac{5b}{9c}\times\frac{28x}{15y}$.

$$\frac{3a}{7b}\times\frac{5b}{9c}\times\frac{28x}{15y}=\frac{a}{1}\times\frac{1}{3c}\times\frac{4x}{3y}=\frac{4ax}{9cy}.$$

EXERCISES. Find the respective products of the following quantities.

1. $-\frac{a^2}{b}$ and $-\frac{b^2}{a}$. Ans. $+ab$.

2 $\frac{a}{b}$, $\frac{b}{c}$, and $\frac{c}{d}$. Ans. $\frac{a}{d}$.

3. $\frac{2x-3y}{2x+3y}$ and $\frac{x+y}{4x-6y}$. Ans. $\frac{x+y}{4x+6y}$.

4. $\frac{a^2-b^2}{a^2+b^2}$ and $\frac{a^3+ab^2}{a^2-2ab+b^2}$. Ans. $\frac{a^2+ab}{a-b}$.

G. $+\frac{5a}{4x}$ and $+\frac{12b}{13x}$.

H. $-\frac{9a}{10b}$ and $+\frac{2c}{3d}$.

K. $+\frac{21mx}{22n^2}$ and $-\frac{7nx}{9m^2}$.

L. $+\frac{14v}{9x}$, $-\frac{5x}{7y}$, and $+\frac{27y}{20z}$.

M. $-\frac{4x^2}{5y}$, $-\frac{15y^2}{16z}$, and $-\frac{36z^2}{35x}$.

N. $\frac{2a-5}{3a-4}$, $\frac{2a+5}{3a+4}$, and $\frac{9a^2-16}{4a^2-25}$.

O. $\frac{10ax-18ay}{7ab-14b^2}$, and $\frac{8a^2-16ab}{15bx-27by}$.

NOTE 2. *When the product of integral quantities and fractions is required*, the former must be multiplied into the numerators of the latter, or previously made into fractions having their denominator 1. Thus, in the following exercise, $3x^2$ may be expressed as $\frac{3x^2}{1}$, and then we may proceed as before.

EXERCISE 5. What is the product of the quantities $3x^2$, $\frac{4a}{x}$, and $\frac{b}{12y}$? Ans. $\frac{abx}{y}$.

6. Of $\frac{2m^2}{33n}$, $\frac{5n^2}{6m}$, and $-18mn$? Ans. $-\frac{10m^2n^2}{11}$.

P. Of $-\frac{3cx}{4by}$, $+20bz$, and $-\frac{6ay}{5cz}$?

7. Of $\frac{a+b}{a^2-2ab+b^2}$, $\frac{a^2+b^2}{a^2-b^2}$, and $(a-b)^3$? Ans. $a^2+b^2$.

NOTE 3. *To multiply a fraction by an integer equal to the denominator of the fraction*, we merely remove the denominator. The numerator, taken as an integer, is the answer.

EXERCISE 8. Multiply $\frac{37ay}{14bx}$ by $14bx$. Ans. $37ay$.

Q. Multiply $\frac{3}{4}$ by 4 and $\frac{x^2}{y}$ by $y$.

R. Multiply $\frac{13a^2+15b^2}{13a^2-15b^2}$ by $13a^2-15b^2$.

NOTE 4. *To multiply a fraction by a whole number when that number is a multiple of the denominator*,—divide the whole number by the denominator, and multiply the numerator by the quotient: the resulting integer is the answer.

The directions given in Notes 3 and 4 require no *Demonstration*, since the process is the same *in substance* as that described in Note 1. These particular cases are here brought prominently into notice from their frequent use in Equations.

EXAMPLE 4. Multiply $\frac{7a}{3x}$ by $12xy$.

$$12xy \div 3x \doteq 4y.$$
$$7a \times 4y \doteq 28ay, \text{ Ans.}$$

EXERCISE 9. Multiply $\frac{3}{4}$ by 12. Ans. 9.

10. Multiply $\frac{3ax}{5b}$ by $35ab$. Ans. $21a^2x$.

11. Multiply $\frac{a+x}{a-x}$ by $a^2-x^2$. Ans. $(a+x)^2$.

S. Multiply $\frac{5}{8}$ by 40, and $\frac{x}{y}$ by $y^2$.

T. What is the product of $\frac{a-b}{13}$ into $39(a+b)$?

U. Multiply $\frac{2y^2-6}{2y^2+6}$ by $4y^4-36$.

## Problem X.

*To find the product of two or more Quantities, one at least of which is a compound Quantity having its terms either all Fractions, or some Fractions, and others Integers.*

**Rule.** We might first reduce the compound quantity or quantities to single terms by addition or subtraction, and then proceed by Problem IX.; but it is generally easier to multiply the terms separately, as directed in Problems II. III. and IV. of Chapter V., assisted by the rule in the last Problem.

**Example 1.** Multiply $a-\frac{b}{2}+\frac{2c}{3}-\frac{3d}{4}+\frac{4e}{5}$ by 12.

$$\begin{array}{l} a-\frac{b}{2}+\frac{2c}{3}-\frac{3d}{4}+\frac{4e}{5} \\ 12 \\ \hline \end{array}$$

Ans. $12a-6b+8c-9d+9\tfrac{3}{5}e$.

**Exercise 1.** Multiply $\frac{a}{3}+\frac{b}{4}-\frac{c}{6}$ by 12.

Ans. $4a+3b-2c$.

2. Multiply $\frac{4x}{5}-\frac{3x}{4}+x-\frac{7x}{3}$ by 60.

Ans. $48x-45x+60x-140x$, or $-77x$.

A. Multiply $\frac{5a}{12}+\frac{3a}{14}-\frac{16a}{21}$ by 84.

B. Multiply $\frac{2w}{15}-\frac{3x}{20}+\frac{7y}{30}-\frac{9z}{40}$ by 60.

**EXAMPLE** 2. Multiply $\frac{15x^2}{16y}-24x+\frac{39y^2}{50x}$ by $\frac{2x}{3y}$.

$$\begin{array}{l}
\frac{15x^2}{16y}-24x+\frac{39y^2}{50x}\\
\frac{2x}{3y}\\
\hline
\text{Ans. } \frac{5x^3}{8y^2}-\frac{16x^2}{y}+\frac{13y}{25}.
\end{array}$$

**EXERCISE** 3. Multiply $a^2-\frac{4ab}{3}+\frac{4b^2}{9}$ by $\frac{3a}{2b}$.

Ans. $\frac{3a^3}{2b}-2a^2+\frac{2ab}{3}$.

C. What is the product of $\frac{y^2}{z^2}-\frac{x^2}{y^2}$ into $\frac{z}{y}$?

D. Of $\frac{4}{9}x^2-xy+\frac{9}{16}y^2$ into $\frac{4y^2}{3x^2}$?

**EXAMPLE** 3. Multiply $x^3-\frac{2}{3}x^2y+\frac{4}{9}xy^2-\frac{8}{27}y^3$ by $x+\frac{2}{3}y$.

$$\begin{array}{llllll}
& x^3-\frac{2}{3}x^2y+\frac{4}{9}xy^2-\frac{8}{27}y^3 & & & & \\
& x+\frac{2}{3}y & & & & \\
\hline
x^4 & -\frac{2}{3}x^3y & +\frac{4}{9}x^2y^2 & -\frac{8}{27}xy^3 & & \\
 & +\frac{2}{3}x^3y & -\frac{4}{9}x^2y^2 & +\frac{8}{27}xy^3 & -\frac{16}{81}y^4. & \\
\hline
\text{Ans. } x^4 & \ldots & \ldots\ \ldots & \ldots & -\frac{16}{81}y^4. &
\end{array}$$

**EXERCISE** 4. Multiply $3a+\frac{4b^2}{5c}$ by $3a-\frac{4b^2}{5c}$.

Ans. $9a^2-\frac{16b^4}{25c^2}$.

5. $\frac{1}{4}m^4+\frac{4}{9}m^2n+\frac{64}{81}n^2$ by $\frac{1}{2}m^2-\frac{8}{9}n$.

Ans. $\frac{1}{8}m^6-\frac{512}{729}n^3$.

E. $\frac{x^2}{y}+x$ by $y+\frac{y^2}{x}$.

F. $\frac{3}{5}a + \frac{5}{3}b$ by $\frac{5}{3}a - \frac{3}{5}b$.

G. $\frac{9a^2}{16} + \frac{6ab}{5} + \frac{16b^2}{25}$ by $\frac{9a^2}{16} - \frac{6ab}{5} + \frac{16b^2}{25}$.

## PROBLEM XI.

*To raise a Fraction to any given Power.*

**RULE.** Raise the terms separately to that power. The *sign* follows the same rule as in integers.

**DEM.** This rule follows from that for the multiplication of fractions.

**EXAMPLE 1.** What is the seventh power of $-\frac{2a^3}{3x^2}$?

$$\left(-\frac{2a^3}{3x^2}\right)^7 = -\frac{128a^{21}}{2187x^{14}}.$$

**EXERCISE 1.** Square $-\frac{7a^5b}{8c^4d^2}$. Ans. $+\frac{49a^{10}b^2}{64c^8d^4}$.

2. What is the square of $\frac{a+5}{a-3}$? Ans. $\frac{a^2+10a+25}{a^2-6a+9}$.

A. What is the fifth power of $-\frac{2ab^2}{3y^3}$?

B. The square of $\frac{12x-15y}{16x-10y}$?

C. The fourth power of $\frac{5+z}{3-z}$?

## PROBLEM XII.

*To raise to a given Power, a compound Quantity whose Terms are either all Fractions or Fractions mixed with Integers.*

**RULE.** Raise it to the given power by actual multiplication, the operation being performed as directed in Problem X.

**EXERCISE 1.** What is the square of $a - \frac{b^2}{2a}$.

Ans. $a^2 - b^2 + \frac{b^4}{4a^2}$.

2. What is the cube of $x + \frac{2}{3}$?

Ans. $x^3 + 2x^2 + \frac{4}{3}x + \frac{8}{27}$.

3. Find the cube of $\frac{3a}{4} - \frac{4b}{5}$.

Ans. $\frac{27a^3}{64} - \frac{27a^2b}{20} + \frac{36ab^2}{25} - \frac{64b^3}{125}$.

A. Find the square of $x + \frac{y}{2}$.

B. The square of $\frac{a}{b} - \frac{b}{c}$.

C. The cube of $2a - \frac{1}{2}b$.

D. The square of $\frac{x^2}{y} + \frac{y^2}{x}$.

E. The square of $\frac{3x}{4} - \frac{4y}{3} + z$.

## PROBLEM XIII.

*To divide one Fraction by another.*

**RULE.** Invert the divisor, and multiply by it when so inverted; or, in other words, multiply the dividend by the reciprocal of the divisor.

*The sign* follows the same rule as in division of integers.

**DEM.** Since $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ by Problem IX., and since Division is the reverse of multiplication, it follows that $\frac{ac}{bd} \div \frac{c}{d} = \frac{a}{b}$. But this is just what we should have obtained by multiplying $\frac{ac}{bd}$ by $\frac{d}{c}$. Now let $ac$ be expressed by the single letter $m$, and $bd$ by $n$, it follows that $\frac{m}{n} \div \frac{c}{d} = \frac{m}{n} \times \frac{d}{c}$.

**EXAMPLE.** Divide $\frac{108a^3b^2c}{7xyz}$ by $\frac{12a^2b}{11z}$.

$$\frac{108a^3b^2c}{7xyz} \div \frac{12a^2b}{11z} = \frac{108a^3b^2c}{7xyz} \times \frac{11z}{12a^2b} = \frac{99abc}{7xy}.$$

**Note.** *If one of the two given quantities—viz. either the divisor or the dividend—is an integer, the other being a fraction,* we proceed in the same manner, first changing the integer into a fraction by giving it the denominator 1.

**Exercise** 1. Divide $\frac{x}{y}$ by $\frac{b}{a}$. Ans. $\frac{ax}{by}$.

2. Divide $\frac{15ab}{13x^2}$ by $\frac{5a}{4x}$. Ans. $\frac{12b}{13x}$.

3. Divide $+\frac{60a^3}{7b}$ by $-9ab$. Ans. $-\frac{20a^2}{21b^2}$.

4. Divide $-9ab$ by $+\frac{60a^3}{7b}$. Ans. $-\frac{21b^2}{20a^2}$.

5. Divide $-\frac{3ac}{5bd}$ by $-\frac{9a}{10b}$. Ans. $+\frac{2c}{3d}$.

A. Divide $+\frac{x}{y}$ by $-\frac{y}{z}$.

B. Divide $-\frac{36a^5b^2}{35xy}$ by $-\frac{54a^3b^3}{49x^2y^2}$.

6. Divide $\frac{a^2+ab}{a-b}$ by $\frac{a^3+ab^2}{a^2-2ab+b^2}$. Ans. $\frac{a^2-b^2}{a^2+b^2}$.

7. Divide $\frac{x+y}{4x+6y}$ by $\frac{2x-3y}{2x+3y}$. Ans. $\frac{x+y}{4x-6y}$.

C. Divide $\frac{45x^2}{a^2-b^2}$ by $\frac{93y^2}{a^2+2ab+b^2}$.

D. Divide $\frac{a^2-5a-14}{a^2+3a-18}$ by $\frac{a-7}{a-3}$.

## Problem XIV.

*To divide one Quantity by another when one of the two is a compound Quantity made up of two or more Terms, all Fractions, or some fractional, others integral; or when both the given Quantities are of that Description.*

**Rule 1.** Reduce the compound quantity, or each of the two compound quantities, to a single fraction (by Pr. II., VII., or VIII., of this Chapter), and then proceed as in the last Problem.

**Rule 2.** Proceed by one of the rules for compound quantities contained in Problems II. and III. for Chapter VI., employing any of the preceding problems in this chapter *for the details.*

**EXAMPLE 1.** Divide $3a-\frac{2ab}{3c}$ by $\frac{a}{c}$.

*By Rule* 1.

$$\left(3a-\frac{2ab}{3c}\right)\div\frac{a}{c}\doteq\frac{9ac-2ab}{3c}\times\frac{c}{a}=\frac{9c-2b}{3},\text{ Ans.}$$

*By Rule* 2.

$$3a\div\frac{a}{c}\doteq\frac{3a}{1}\times\frac{c}{a}=3c.$$

$$\frac{2ab}{3c}\div\frac{a}{c}\doteq\frac{2ab}{3c}\times\frac{c}{a}=\frac{2b}{3}.$$

$$\text{Ans. } 3c-\frac{2b}{3}.$$

**EXERCISE 1.** Divide $5a-\frac{7a^2}{8b}$ by $\frac{35a}{b}$, by both rules.

Ans. $\frac{40b-7a}{280}$, or $\frac{b}{7}-\frac{a}{40}$.

2. Divide $\frac{4x^2z}{y}-\frac{8xz}{15}+\frac{3xz^2}{y}$ by $\frac{4xz}{5y}$, by Rule 2.

Ans. $5x-\frac{2y}{3}+\frac{15z}{4}$.

A. Divide $x-\frac{y}{a}$ by $\frac{x}{a}$, by both Rules.

B. Divide $3x-\frac{7y}{10}$ by 21, by Rule 1.

C. Divide $\frac{3}{4}y-\frac{5}{8}z$ by $\frac{15y}{16z}$, by both Rules.

D. Divide $a^3-\frac{2a^2}{3b}+\frac{3a}{4b^2}$ by $\frac{2a}{5b}$, by Rule 2.

**EXAMPLE 2.** Divide $a^2-\frac{b^2}{c^2}$ by $a+\frac{b}{c}$, by Rule 1.

$$\left(a^2-\frac{b^2}{c^2}\right)\div\left(a+\frac{b}{c}\right)\doteq\frac{a^2c^2-b^2}{c^2}\div\frac{ac+b}{c}=\frac{a^2c^2-b^2}{c^2}\times\frac{c}{ac+b}$$

$$=\frac{ac-b}{c}=a-\frac{b}{c}.$$

**EXERCISE 3.** Divide $9a^2-\frac{16b^4}{25c^2}$ by $3a-\frac{4b^2}{5c}$, by Rule 1.

Ans. $\frac{15ac+4b^2}{5c}$, or $3a+\frac{4b^2}{5c}$.

E. Divide $a^2-b^2+\frac{b^4}{4a^2}$ by $a-\frac{b^2}{2a}$, by Rule 1.

**Example** 3. Divide $\frac{1}{8}m^6-\frac{512}{729}n^3$ by $\frac{1}{4}m^4+\frac{4}{9}m^2n+\frac{64}{81}n^2$, by Rule 2.

$$\begin{array}{l}
\tfrac{1}{4}m^4+\tfrac{4}{9}m^2n+\tfrac{64}{81}n^2\,)\;\tfrac{1}{8}m^6\;\ldots\;\ldots\;\ldots\;-\tfrac{512}{729}n^3\,(\;\tfrac{1}{2}m^2-\tfrac{8}{9}n. \\
\qquad\qquad\qquad\qquad\qquad \underline{\tfrac{1}{8}m^6+\tfrac{2}{9}m^4n+\tfrac{32}{81}m^2n^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad -\tfrac{2}{9}m^4n-\tfrac{32}{81}m^2n^2-\tfrac{512}{729}n^3 \\
\qquad\qquad\qquad\qquad\qquad\qquad -\tfrac{2}{9}m^4n-\tfrac{32}{81}m^2n^2-\tfrac{512}{729}n^3.
\end{array}$$

**Exercise** 4. Divide $x^4-\frac{16}{81}y^4$ by $x+\frac{2}{3}y$, by Rule 2.

Ans. $x^3-\frac{2}{3}x^2y+\frac{4}{9}xy^2-\frac{8}{27}y^3$.

F. Divide $\frac{a^2}{b^2}-2\frac{a}{c}+\frac{b^2}{c^2}$ by $\frac{a}{b}-\frac{b}{c}$, by Rule 2.

G. Divide $\frac{27a^3}{64}-\frac{27a^2b}{20}+\frac{36ab^2}{25}-\frac{64b^3}{125}$ by $\frac{3a}{4}-\frac{4b}{5}$, proceeding by Rule 2.

## PROBLEM XV.

*To find any proposed Root of any given Fraction.*

**Rule.** Extract the proposed root of each term, separately.

The *sign* follows the same rule as in integers.

**Dem.** The rule is the direct consequence of that for Problem xi.

**Example** 1. What is the fourth root of $\frac{16x^4}{81y^{12}}$?

$$\sqrt[4]{\frac{16x^4}{81y^{12}}}=\pm\frac{2x}{3y^3}.$$

**Exercise** 1. What is the cube root of $-\frac{8x^3y^6}{125a^9}$?

Ans. $-\frac{2xy^2}{5a^3}$.

2. What is the square root of $\frac{a^2+10a+25}{a^2-6a+9}$?

Ans. $\pm\frac{a+5}{a-3}$.

A. Find the square root of $\frac{452929a^{10}}{149769b^{12}}$.

B. The cube root of $+\frac{125a^3b^6c^{15}}{343c^9d^{12}}$.

C. The square root of $-\frac{64a^{10}}{144x^{20}}$.

D. The square root of $\frac{144x^2-360xy+225y^2}{256x^2+320xy+100y^2}$.

## PROBLEM XVI.

*To find the Square Root of a compound Quantity consisting of fractional (or fractional mixed with integral) Terms.*

**RULE.** Proceed as directed in Problem II. of Chapter VIII., with the assistance of the various preceding problems in this chapter.

**EXAMPLE.** Find the square root of $\frac{25x^2}{64y^2}-\frac{9}{8}+\frac{81y^2}{100x^2}$.

$$\begin{array}{l|l}
\frac{5x}{8y} & \frac{25x^2}{64y^2}-\frac{9}{8}+\frac{81y^2}{100x^2}\left(\frac{5x}{8y}-\frac{9y}{10x},\ \text{Ans.}\right. \\
\frac{5x}{8y} & \underline{\frac{25x^2}{64y^2}} \\
\hline
\frac{5x}{4y}-\frac{9y}{10x} & -\frac{9}{8}+\frac{81y^2}{100x^2} \\
 & \underline{\underline{-\frac{9}{8}+\frac{81y^2}{100x^2}.}}
\end{array}$$

**EXERCISE 1.** What is the square root of $a^6-\frac{3}{4}a^4b^2+\frac{9}{64}a^2b^4$? Ans. $a^3-\frac{3}{8}ab^2$.

2. Of $\frac{1}{4}x^2-\frac{2}{3}x^3+\frac{43}{36}x^4-x^5+\frac{9}{16}x^6$?

Ans. $\frac{1}{2}x-\frac{2}{3}x^2+\frac{8}{4}x^3$.

A. Of $\frac{a^2}{b^2}+2\frac{a}{c}+\frac{b^2}{c^2}$?

B. Of $\frac{4}{9}a^2-ab+\frac{9}{16}b^2$?

C. Of $1+\frac{x}{y}-\frac{5x^2}{4y^2}-\frac{3x^3}{4y^3}+\frac{9x^4}{16y^4}$?

# CHAPTER XII.

## EQUATIONS.

### DEFINITIONS AND EXPLANATIONS.

1. When two quantities are equal, the expression of that equality is called an *Equation.*

2. When two quantities are *necessarily* equal (that is, when they are equal under every value that can be assigned to each letter, or when they consist of equal combinations of particular numbers), the quantities are said to be *Identical,* and sometimes the equation itself, expressing their equality, is designated by the same word. Thus $6+4=2\times 5$, and $(a+x)\times(a-x)=a^2-x^2$, are identical equations, or equations of identical quantities.

3. An equation is said to be *Impossible* when no possible value can be given to the unknown quantity which will satisfy its conditions; as $\sqrt{x^2+1}-\sqrt{x^2-1}=0$, or $y^2-y+10=0$.

4. When an equation (not identical or impossible) contains an unknown quantity mixed with known quantities (as $3x^2-5x=2$), its value is said to be *implicitly* expressed by the equation: that is, the equation contains enough to render the value of the unknown quantity determinate, although it does not directly inform us what that value is. But when a quantity previously unknown is brought by any means to stand alone on the one side, and known quantities only on the other (as $x=a+b$), we then have the value of the previously unknown quantity *explicitly* declared.

5. Equations are applied to various uses; but their peculiar purpose, when treated as a separate branch of Algebra, is to enable us to obtain explicitly the values of unknown quantities, from given equations in which those values are only implicitly expressed. Thus, if we have given the equation, $\frac{1}{2}x+\frac{1}{3}x=10$, that equation, after certain operations, is transformed into $x=12$. We are then said to have *solved* or *resolved* the given equation; and the process by which we do so, is called the *Solution* or *Resolution of Equations.*

6. When equations are thus used for the determination

of unknown quantities, they are distinguished by different names, according to the highest power of any unknown quantity involved in them. If an equation, when cleared of fractions and roots, contains no power of any unknown quantity higher than the first, it is called a *Simple Equation;* if none higher than the second, a *Quadratic Equation;* if it contains the third power and no higher, it is called a *Cubic Equation;* and if the fourth power and no higher, a *Biquadratic Equation.*

7. These are also designated by the expressions—*Equations of the first, second, third, and fourth Degrees;* and, in general, an equation is said to be of that degree which is indicated by the highest power of the unknown quantity contained in it. Thus $x^5+4x^3+6=70$ is called an equation of the fifth degree.

8. Simple and quadratic equations are usually (and those of higher degrees occasionally) resolved by successive applications of the following general principles, some of which are self-evident, and others derived from those which are so, or, in some instances, mere repetitions of them in other forms, one form being sometimes more convenient, sometimes another.

### GENERAL PRINCIPLES.

1. Equal quantities* may be added to both sides.
2. Equal quantities* may be taken from both sides.
3. Both sides may be multiplied by equal quantities.*
4. Both sides may be divided by equal quantities.*
5. Both sides may be raised to the same power.
6. The same root of both sides may be extracted.
7. For any quantity its equal may be substituted.
8. Any term may be transposed from the one side to the other, if its sign be changed.
9. *All* the terms on *both* sides may be transposed without changing any of the signs.
10. *All* the signs of *both* sides may be changed, without transposing any of the terms.

Any of these operations may be performed, and the equality will still be preserved.

The first four principles are self-evident. The mark =, indicating equality, appears to have been originally the emblem of a balance, being still found, in that capacity, in a modified form, among the marks used for the signs of the zodiac; and nothing affords a better illustration, than a balance, of these four rules.

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* Or the same quantity.

The fifth principle is derived from the third; for, when two equals are raised both to the same power, they are, at each successive step, multiplied by equals: consequently the successive products are equal, including the final product.

The sixth follows from the fifth.

The seventh is self-evident.

The eighth is derived from the first and second; for, if $a-b \doteq c$, add $b$ to each side, and $a \doteq c+b$; or, if $a+b \doteq c$, take $b$ from each side, and $a \doteq c-b$.

The ninth principle is self-evident; for, if $a \doteq b$, $b \doteq a$, whether $a$ and $b$ be positive or negative, and whether simple or compound quantities.

The tenth may also be regarded as self-evident; for, if $+a \doteq +b$, $-a \doteq -b$. But it may be deduced from the eighth and ninth if we choose: for we may transpose each term, changing the sign by Prin. VIII., and then re-transpose all the terms without changing the signs, by Prin. IX. Thus, if $a-b \doteq x-y$, then $-x+y \doteq -a+b$ by Prin. VIII., and $-a+b \doteq -x+y$, by Prin. IX.

The OBJECTS to be attained by the application of these principles, are—

1*st.* To bring all the terms containing unknown quantities to one side (that side being almost invariably the left), and all the other terms to the other side.

2*d.* To produce an equation containing only one unknown quantity.

3*d.* To clear the unknown quantity of all co-efficients, divisors, and exponents of powers or roots, so that, at last, it may stand out alone on the one side, and its value in known quantities on the other.

These objects are not always to be attained in the order here described, but sometimes in one order, at other times in another, according to circumstances.

In the case of simple equations with only one unknown quantity, the circumstances and modes, in which the different principles should be employed, will be almost obvious if we keep in mind the *objects* to be attained; but they will be still better understood from the examples. In the cases of equations involving more than one unknown quantity, and of quadratic and higher equations, *particular rules* become necessary.

When there is only one unknown quantity, one equation is always sufficient to determine its value, unless it be an identical equation: but if there are more unknown quantities than one, there must be as many independent equations.

When the value of an unknown quantity is found, it is often useful to *prove* the result, by inserting that value, instead of the unknown quantity, in the original equation. If the value found is correct, the equation becomes identical.

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# CHAPTER XIII.

## SIMPLE EQUATIONS.

### PROBLEM I.

*To resolve a simple Equation containing one unknown Quantity.*

RULE. Apply one or more of the *General Principles* given in the last chapter, in such a way as to attain the *Objects* there specified. When at last the unknown quantity has been brought to stand alone on the left side, and its value, in known quantities, on the right, the equation is resolved.

EXAMPLE 1. Having given $x-5=7$, to find the value of $x$.

Adding 5 to each side, according to the first General Principle, we have

$$x=12.$$

PROOF. $12-5 \doteq 7$.

NOTE 1. In subsequent examples it will be unnecessary to give the proof; but it may be useful for the learner to apply it in many, or all, of the exercises.

EXERCISE 1. Given $x-17=24$, to determine $x$.

Ans. $x=41$.

A. Given $y-59=75$, to find $y$.

B. ... $z-36=-19$, to find the value of $z$.

NOTE 2. In the following examples and exercises, when we say merely that a certain equation is given, it is understood that we are to determine the value or values of the unknown quantity or quantities expressed by the letters $x$, $y$, $z$.

EXAMPLE 2. Given $x + b = a$.

Subtracting $b$ from each side by Principle 2, or transposing it from the left to the right by Prin. 8, we have

$$x = a - b.$$

EXERCISE 2. Given $x + 15 = 20$. Ans. $x = 5$.
3. Given $x + 3c = 5b$. Ans. $x = 5b - 3c$.
C. ... $y + 48 = 94$.
D. ... $z - 3a + 15 = 8b$.

EXAMPLE 3. Given $5a - z = 2b$.

By Prin. 8, $-z \doteq -5a + 2b$.
Then, by Prin. 10, $z \doteq +5a - 2b$.

EXERCISE 4. Given $128 - x = 46$. Ans. $x = 82$.
E. Given $36 - y = -15$.
F. ... $654 - z = 308$.

EXAMPLE 4. Given $ax = c$.

Dividing both sides by $a$, by Prin. 4,

$$x \doteq \frac{c}{a}.$$

EXERCISE 5. Given $52x = 130$. Ans. $x = 2\frac{1}{2}$.
G. Given $7x = 35$.
H. ... $286y = 2574$.
I. ... $93z = 1441\frac{1}{2}$.
J. ... $mx = na$.

EXAMPLE 5. Given $\frac{x}{a} = b - c$, or $x \div a = b - c$.

Multiply both sides by $a$, by Prin. 3, the fraction being multiplied according to the rule in Note 3 of Pr. IX. Ch. XI. Then we have

$$x = ab - ac.$$

EXERCISE 6. Given $\frac{x}{5} = 8$. Ans. $x = 40$.
7. Given $\frac{y}{b} = a + b$. Ans. $y = ab + b^2$.
K. ... $x \div 13 = 16$.
L. ... $64z \div 25 = 320$.

EXAMPLE 6. Given $125 - \frac{3x}{4} = 41$.

Proceeding by successive operations, we have—

Taking 41 from both sides, $84 - \frac{3x}{4} = 0$.
Multiplying by 4, $336 - 3x = 0$.
Transposing 336, $-3x = -336$.
Changing all the signs, $3x = 336$.
Dividing by 3, $x = 112$.

EXERCISE 8. Given $4x+56=17x-35$. Ans. $x=7$.

9. Given $\frac{5x}{12}=45$. Ans. $x=108$.

10. ... $2y-3a=10b$. Ans. $y=\frac{3}{2}a+5b$.

11. ... $\frac{7z}{10}-63=-42$. Ans. $z=30$.

12. ... $3a-\frac{3x}{20}=8c$. Ans. $x=20a-53\frac{1}{3}c$.

13. ... $\frac{6z-4}{3}-5=\frac{18-4z}{3}+z$. Ans. $z=5\frac{2}{7}$.

M. ... $12y+74=20y-26$.

N. ... $18x-252=-132$.

O. ... $\frac{3x}{5}-6a=15b$.

P. ... $\frac{7x-10}{23}+x=\frac{109-5x}{23}+7$.

NOTE 3. When several fractions occur with different denominators, and all involving the unknown quantity, the fractions may be changed into whole numbers either by multiplying by their denominators successively, or by multiplying at once by any common multiple of all the denominators. The shortest mode, and perhaps the easiest, is to multiply at once by the *least* common multiple of the whole, observing the rule contained in Note 4 to Problem IX. of Chapter XI.

EXAMPLE 7. Given $\frac{x}{2}+\frac{x}{3}-\frac{x}{5}=38$.

Here the least common multiple of all the denominators is their product 30. We therefore multiply at once by 30, by Prin. 3, and find

$$15x+10x-6x=1140.$$

That is, $19x=1140$.

By Prin. 4, $x=60$.

EXERCISE 14. Given $\frac{x}{2}+\frac{2x}{3}=147$. Ans. $x=126$.

15. Given $\frac{y+2}{5}=\frac{2y-6}{3}-6$. Ans. $y=18$.

Q. ... $\frac{2x}{3}-\frac{3x}{4}+x=11$.

R. ... $\frac{2x}{5}+\frac{x}{6}-\frac{3x}{7}=10$.

**Example** 8. Given $\frac{3z}{4}-\frac{z}{6}+\frac{5z}{9}=100$.

The least common multiple of 4, 6, and 9 being 36, we multiply all the terms by that, and we have

$$27z-6z+20z=3600,$$
$$\text{Or}\qquad 41z=3600.$$
$$\therefore,\ \text{by Prin. 4},\ z\doteq 87\tfrac{33}{41}.$$

**Exercise** 16. Given $\frac{5x}{6}-\frac{7x}{10}-64=0$. Ans. $x=480$.

17. Given $\frac{x}{6}-\frac{x}{4}-9=\frac{x}{3}-\frac{x}{2}$. Ans. $x=108$.

18. ... $\frac{20x}{21}-75=\frac{5x}{12}+\frac{5x}{16}$. Ans. $x=336$.

S. ... $\frac{7x}{8}-\frac{5x}{6}=1$.

T. ... $\frac{x}{12}+\frac{x}{15}-\frac{x}{20}=9$.

U. ... $\frac{z}{2}-\frac{z}{3}+\frac{z}{4}-\frac{z}{5}+1=27$.

V. ... $\frac{5y-5}{12}+\frac{3y+1}{10}=\frac{y}{2}+2\tfrac{1}{2}$.

W. ... $\frac{34y-5}{20}+\frac{18y+3}{16}=\frac{42y}{15}$.

19. ... $\frac{30}{x}=\frac{40}{x}-\frac{5}{6}$. Ans. $x=12$.

20. ... $\frac{y}{4-y}-7=24$. Ans. $y=3\tfrac{7}{8}$.

X. ... $8-\frac{35}{z}=\frac{28}{2z}+\frac{21}{3z}$.

Y. ... $\frac{x}{x+5}+\frac{1}{5}=1$.

**Note** 4. When a fraction, having its numerator a compound quantity, is preceded by a negative sign, we must either transpose the whole fraction, changing its sign, or we must keep in mind that, in reducing it to an integral form, the vinculum is removed (See Chap. I. 33), and must therefore change all the signs of the numerator (See Chap. IV. Remark 4).

**Example** 9. Given $6x-\frac{2x-3}{5}=51$.

Transposing the fraction, $6x \doteq 51 + \frac{2x-3}{5}$.

Multiplying by 5, $30x \doteq 255 + 2x - 3$.

Hence $28x \doteq 252$, and $x = 9$.

*Or thus*:—Multiplying by 5 without transposing, and observing Remark 4 of Ch. IV.,

$$30x - 2x + 3 \doteq 255.$$

Hence $28x \doteq 252$, and $x = 9$.

EXERCISE 21. Given $\frac{5x}{2} - \frac{3x+5}{4} = 11$. Ans. $x = 7$.

22. Given $5x + \frac{7x+2}{3} - \frac{5x-6}{7} = 28$. Ans. $x = 4$.

23. ... $\frac{3x-33}{4} - 12 = \frac{21-x}{2} - \frac{8x+45}{6}$. Ans. $x = 9$.

24. ... $\frac{3x-6}{4} - \frac{3x-8}{3} = 10\frac{2}{3} - \frac{54+4x}{9}$.

Ans. $x = 18$.

Z. ... $\frac{z-7}{2} - \frac{z+7}{3} + \frac{z}{9} = 0$.

AA. ... $\frac{22-3z}{6} - \frac{76-9z}{14} = -\frac{6z-11}{21}$.

BB. ... $\frac{5x-1}{2} - \frac{3x+15}{8} = 6x - \frac{7x-3}{6} - 11$

CC. ... $\frac{1}{5-x} - \frac{1}{5+x} = \frac{2}{25-x^2}$.

## PROBLEM II.

*To find the Values of two unknown Quantities, from two independent Equations.*

RULE. Having simplified and arranged each equation, if necessary, by changing fractional quantities into integers, and collecting all the terms containing one of the unknown quantities into one term, all those containing the other unknown quantity into another term, and all the known quantities into a single group, as in the following examples, and by processes exactly the same as those in the preceding problem, each equation will then appear in the form, $ax + by = c$, in which $a$, $b$, and $c$ may be any known quantities whatever, simple or compound. We then, from the two equations combined, produce one resulting equation containing only one unknown quantity, from which the value of that quantity is found by Problem I. This resulting equation may be obtained by any one of the three methods presently to be described.

When the value of the one unknown quantity has been found, it is inserted, instead of that quantity, in either of the two original simplified equations, and we obtain another equation, containing only the other unknown quantity, also to be resolved by Problem I.

METHOD FIRST. In either equation treat one of the unknown quantities as if it were known, and find the value of the other in terms of the first and of known quantities. Having found that value, insert it, instead of its equivalent letter, in the other equation. That equation will then contain only one unknown quantity.

METHOD SECOND. From the two equations, treated separately, find two separate values for one of the two unknown quantities in terms of the other and of known quantities, in precisely the same manner as we found one value from one equation by Method First. The two values thus found being, of necessity, equal, give us a resulting equation containing only one unknown quantity.

METHOD THIRD. Multiply each side of both equations by such numbers as will equalize the two co-efficients of one of the unknown quantities. We have thus two new equations, the sum or difference of which is our resulting equation containing only one unknown quantity.

NOTE 1. The quantity which disappears in the resulting equation, is said to be *eliminated.*

NOTE 2. In following the third method, it may be that the two co-efficients of one of the unknown quantities in the given equations are equal at first (as in Example 2): in that case no multiplication is necessary. Or one of the given co-efficients may be a multiple of the other (as in Example 3); in that case only one equation requires to be multiplied. *In general,* the two co-efficients will be rendered equal if we use each as a multiplier of the other equation (as in Example 1): but the operation is facilitated if we previously divide the co-efficients by their greatest common measure (if they have a common measure greater than 1), and use the quotients as the multipliers (as in Example 4): in this case the two co-efficients obtained by the multiplication are each the least common multiple of the two original co-efficients. Similar modifications may be made in Methods First and Second.

### EXAMPLE 1.

Given $\left\{\begin{array}{l} 2x-3y=\ 7, \\ 3x-4y=12, \end{array}\right\}$ to find $x$ and $y$.

*By Method First.*

From Equation 1[*], $x=\dfrac{7+3y}{2}$.

Inserting this value for $x$ in Equation 2,

$$3\times\frac{7+3y}{2}-4y=12.$$

That is, $\dfrac{21+9y}{2}-4y=12$.

Or $21+9y-8y=24$.

$$\therefore y=3.$$

*By Method Second.*

From Eq. 1, $x=\dfrac{7+3y}{2}$.

From Eq. 2, $x=\dfrac{12+4y}{3}$.

$$\therefore \frac{7+3y}{2}=\frac{12+4y}{3}.$$

Multiplying by 6, $21+9y=24+8y$.

Hence $y=3$.

*By Method Third.*

Multiply Eq. 1 by 3. Then $6x-9y=21$.

Multiply Eq. 2 by 2. $6x-8y=24$.

Subtract the first of the two new equations from the second. Then

$$y=3.$$

Having now found $y=3$ by each of the three methods, the remainder of the process is the same for all three: that is, we insert 3 for $y$ in either of the two given equations—say in the first, and we have the resulting equation,

$$2x-9=7.$$

Hence $x=8$.

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[*] That is, the first of the two given equations.

**Example 2.** Given $\left\{\begin{array}{l}4x+3y=\ \ \ 51,\\4x-5y=-21,\end{array}\right\}$ to determine $x$ and $y$ by Method Third.

Subtracting Eq. 2 from Eq. 1, $8y \doteq 72$.

Hence we find $y=9$, and $x=6$.

**Example 3.** Given $\left\{\begin{array}{l}5x-\frac{3}{2}y=\ \ 39,\\10x-\frac{4}{7}y=112,\end{array}\right\}$ to determine $x$ and $y$ by Method Third.

Doubling Eq. 1, $10x-3y \doteq 78$.

Subtract this from Eq. 2. $3y-\frac{4}{7}y \doteq 34$.

Hence $y \doteq 14$, and $x \doteq 12$.

**Example 4.** Given $\left\{\begin{array}{l}12x+\ \ 3y=114,\\8x+10y=156,\end{array}\right\}$ to find $x$ and $y$ by Method Third.

The greatest common measure of 12 and 8 is 4. $12 \div 4 \doteq 3$; and $8 \div 4 \doteq 2$. Therefore we multiply the two given equations respectively by 2 and 3, and obtain

$$24x+6y=228,$$
$$24x+30y=468.$$

By subtracting, $24y \doteq 240$.

Hence, $y \doteq 10$, and $x=7$.

**Note 3.** The third method of operation is usually the most pleasant, from the absence of fractions. It is also frequently superior to the others in point of facility, and seldom inferior. It is therefore the best mode, although the others may be practised.

**Example 5.** Given $\left\{\begin{array}{l}13x-4y=14,\\15x+2y=36,\end{array}\right\}$ to find the values of $x$ and $y$ by Method Third.

Here we find it the easiest mode to eliminate $y$. We therefore multiply the second equation by 2, and *add* the product to the first. We then have

$$43x=86.$$

Hence $x \doteq 2$, and $y=3$.

**Note 4.** In peculiar circumstances we may occasionally deviate a little from the direct process by either of the three methods—sometimes with advantage. Thus, returning to Example 1, if we had begun by subtracting Eq. 1 from Eq. 2, we should have had $x-y=5$, and $x=y+5$, which would have been a simpler value to substitute for $x$ by Method First, or to use as one of the two values of $x$ by Method Second. We might also, if we had chosen, have used it instead of Eq. 2, in following the Third Method.

**Exercise 1.** Given $\left\{\begin{array}{l}x+y=25,\\x-y=\ \ 7,\end{array}\right.$ Ans. $\left\{\begin{array}{l}x=16,\\y=\ \ 9.\end{array}\right.$

2. Given $\begin{cases} 6x+4y=72, \\ 6x-3y=30. \end{cases}$ Ans $\begin{cases} x=8, \\ y=6. \end{cases}$

A. ... $\begin{cases} 2x+3y=33, \\ 2x-3y=\phantom{0}3. \end{cases}$

3. ... $\begin{cases} 3x+7y=117, \\ 12x-5y=\phantom{0}72. \end{cases}$ Ans. $\begin{cases} x=11, \\ y=12. \end{cases}$

B. ... $\begin{cases} 4x+5y=128, \\ 2x+3y=\phantom{0}72. \end{cases}$

C. ... $\begin{cases} 11y-4z=\phantom{0}76, \\ 15y-4z=108. \end{cases}$

4. ... $\begin{cases} 6x-7y=-9, \\ 7x-6y=-4. \end{cases}$ Ans. $\begin{cases} x=2, \\ y=3. \end{cases}$

5. ... $\begin{cases} 10x+13y=36, \\ 15x+19y=53. \end{cases}$ Ans. $\begin{cases} x=1, \\ y=2. \end{cases}$

6. ... $\begin{cases} 13x-2y=58, \\ 15x-2y=70. \end{cases}$ Ans. $\begin{cases} x=\phantom{0}6, \\ y=10. \end{cases}$

7. ... $\begin{cases} 7x+3y=122, \\ 9x+\phantom{3}y=144. \end{cases}$ Ans. $\begin{cases} x=15\tfrac{1}{2}, \\ y=\phantom{0}4\tfrac{1}{2}. \end{cases}$

8. ... $\begin{cases} 73x+4y=446, \\ 50x-6y=288. \end{cases}$ Ans. $\begin{cases} x=6, \\ y=2. \end{cases}$

D. ... $\begin{cases} 6x-5z=12, \\ 4x+3z=15. \end{cases}$

E. ... $\begin{cases} 7x-\phantom{1}2z=134, \\ 5x-11z=\phantom{13}0. \end{cases}$

F. ... $\begin{cases} 16x-3y=32, \\ 20x+6y=53. \end{cases}$

G. ... $\begin{cases} 7x+\phantom{2}y=\phantom{1}71, \\ 8x+2y=112. \end{cases}$

H. ... $\begin{cases} 2\tfrac{1}{2}x+3\tfrac{1}{3}y=325, \\ 3\tfrac{1}{3}x+2\tfrac{1}{2}y=305. \end{cases}$

9. ... $\begin{cases} 7\tfrac{1}{2}y+13\tfrac{1}{3}z=660, \\ 8\tfrac{1}{3}y+\phantom{1}5\tfrac{1}{2}z=398. \end{cases}$ Ans. $\begin{cases} y=24, \\ z=36. \end{cases}$

10. ... $\begin{cases} x+y=20, \\ x\div y=\tfrac{2}{3}. \end{cases}$ Ans. $\begin{cases} x=\phantom{1}8, \\ y=12. \end{cases}$

11. ... $\begin{cases} x^2-y^2=21, \\ x\ -y\ =\ 1. \end{cases}$ Ans. $\begin{cases} x=11, \\ y=10. \end{cases}$

H. ... $\begin{cases} 2\tfrac{1}{2}x+3\tfrac{1}{3}y=325, \\ 3\tfrac{1}{3}x+2\tfrac{1}{2}y=305. \end{cases}$

K. ... $\begin{cases} y-z=2\tfrac{1}{4}, \\ y\div z=2\tfrac{1}{2}. \end{cases}$

L. ... $\begin{cases} x^2-z^2=95, \\ x\ +z\ =19. \end{cases}$

12. ... $\begin{cases} \dfrac{2x+3y}{6}+\dfrac{x}{3}=8, \\ \dfrac{7y-3x}{2}-y=11. \end{cases}$ Ans. $\begin{cases} x=6, \\ y=8. \end{cases}$

13. Given $\begin{cases} \dfrac{7+x}{5} - \dfrac{2x-z}{4} = 3z - 5, \\ \dfrac{5z-7}{2} + \dfrac{4x-3}{6} = 18 - 5x. \end{cases}$ Ans. $\begin{cases} x = 3, \\ z = 2. \end{cases}$

M. ... $\begin{cases} \dfrac{7x-9y}{3} = 8, \\ \dfrac{5x+5y}{4} = 16. \end{cases}$

N. ... $\begin{cases} \dfrac{x+y}{2} + \dfrac{x-y}{3} = 7, \\ \dfrac{2x+3y}{4} - \dfrac{3x+2y}{6} = \dfrac{x}{4}. \end{cases}$

O. ... $\begin{cases} \dfrac{3x+4z}{30} - \dfrac{5x+2z}{40} = 8\frac{4}{5}, \\ \dfrac{4x-z}{14} - \dfrac{2x-3z}{4} = 70\frac{1}{14}. \end{cases}$

### PROBLEM III.

*To find the Values of three or more unknown Quantities from as many independent Equations.*

**RULE.** There are three methods of proceeding, corresponding to the three methods of the last Problem, by any one of which a new set of equations is produced, one fewer in number, and entirely cleared from one of the unknown quantities. By repeating the same process with the new set of equations, another of the unknown quantities is eliminated; and so on till only one remains, the value of which we find by Problem I.

**NOTE 1.** Peculiar facilities will often be found in particular cases, dispensing with the necessity of going through the whole process, as in Exercises 1, A, and B.

**EXAMPLE.** Given $\left\{\begin{array}{l} x + y + z = +6, \\ 2x + 3y - 4z = -4, \\ 3x + 8y - 6z = +1, \end{array}\right\}$ to find $x$, $y$, and $z$.

*By Method First.*

From Eq. 1, $x = 6 - y - z$.

Inserting this value in Eqs. 2 and 3, we have

$$2(6 - y - z) + 3y - 4z = -4,$$
$$\text{and } 3(6 - y - z) + 8y - 6z = +1.$$

These two equations, simplified, become

$$y - 6z = -16, \text{ and}$$
$$5y - 9z = -17.$$

We have thus found two equations containing only $y$ and $z$.

### *By Method Second.*

From the three given equations, separately, we obtain

$$x = 6 - y - z,$$
$$x = \frac{-4 - 3y + 4z}{2},$$
$$x = \frac{1 - 8y + 6z}{3}.$$

And, equating these with each other, two by two,

$$\text{we have } 6 - y - z = \frac{-4 - 3y + 4z}{2},$$
$$\text{and} \qquad 6 - y - z = \frac{1 - 8y + 6z}{3}.$$

These two equations, simplified, and cleared of fractions, become

$$y - 6z = -16, \text{ and}$$
$$5y - 9z = -17, \text{ as before.}$$

### *By Method Third.*

$$\text{From Eq. 1, } 2x + 2y + 2z \doteq 12,$$
$$\text{and} \qquad 3x + 3y + 3z \doteq 18.$$

Subtract these, severally, from Eqs. 2 and 3. Then

$$y - 6z \doteq -16, \text{ and}$$
$$5y - 9z \doteq -17, \text{ as before.}$$

Having thus, by each of the three different methods, obtained the same two resulting equations containing only $y$ and $z$, we proceed to find the values of $y$ and $z$ by Problem II.,—viz. 2 and 3,—and then, inserting these values for $y$ and $z$ in Eq. 1, we obtain another resulting equation containing only $x$, viz. $x + 5 = 6$, from which we find $x = 1$.

In working this example we might have deviated from the direct process with advantage, by first subtracting Eq. 1 from Eq. 2, and then Eq. 2 from Eq. 3. The two new equations, along with Eq. 1, would have been more easily treated by any of the three methods, than our three original equations.

EXERCISE 1. Given $\left\{\begin{aligned} x+y+z&=22,\\ x+y-z&=20,\\ x-y+z&=4.\end{aligned}\right.$ Ans. $\left\{\begin{aligned} x&=12,\\ y&=9,\\ z&=1.\end{aligned}\right.$

2. Given $\left\{\begin{aligned} 2x+2y-3z&=18,\\ 3x-2y+z&=20,\\ 4x-4y-z&=2.\end{aligned}\right.$ Ans. $\left\{\begin{aligned} x&=10,\\ y&=8,\\ z&=6.\end{aligned}\right.$

A. ... $\left\{\begin{aligned} x+y+z&=19,\\ x-y-z&=5,\\ x+2y+3z&=28.\end{aligned}\right.$

3. ... $\left\{\begin{aligned} x-y&=2,\\ y+z&=9,\\ x-z&=4.\end{aligned}\right.$ Ans. $\left\{\begin{aligned} x&=7\tfrac{1}{2}.\\ y&=5\tfrac{1}{2}.\\ z&=3\tfrac{1}{2}.\end{aligned}\right.$

4. ... $\left\{\begin{aligned} x-y+z&=8,\\ x+2y+3z&=84,\\ \tfrac{1}{2}x+\tfrac{1}{3}y-\tfrac{1}{4}z&=-1.\end{aligned}\right.$ Ans. $\left\{\begin{aligned} x&=0,\\ y&=12,\\ z&=20.\end{aligned}\right.$

B. ... $\left\{\begin{aligned} \tfrac{1}{2}x+\tfrac{1}{3}y+\tfrac{1}{4}z&=10,\\ x+y+z&=30,\\ 2x+3y+4z&=100.\end{aligned}\right.$

C. ... $\left\{\begin{aligned} 5x-4y+2z&=+21,\\ 3x+2y-2z&=+17,\\ 7x-8y-6z&=-89.\end{aligned}\right.$

5. ... $\left\{\begin{aligned} \tfrac{1}{3}(x-y)+z&=22,\\ x+\tfrac{1}{2}(y+z)&=16,\\ (x-y+z)\div z&=4.\end{aligned}\right.$ Ans. $\left\{\begin{aligned} x&=+18,\\ y&=-15,\\ z&=+11.\end{aligned}\right.$

D. ... $\left\{\begin{aligned} \tfrac{1}{5}(x+y)-\tfrac{1}{8}z&=2\tfrac{1}{2},\\ \tfrac{1}{9}(y+z)-\tfrac{1}{6}x&=2\tfrac{1}{2},\\ \tfrac{1}{3}(x+z)-\tfrac{1}{4}y&=5\tfrac{2}{3}.\end{aligned}\right.$

# CHAPTER XIV.

## QUESTIONS PRODUCING SIMPLE EQUATIONS.

The method of proceeding, in finding answers to questions by Algebra, is, to represent the unknown quantities by one or more of the letters $x$, $y$, $z$, and then to express, in one or more equations, the different facts or suppositions stated regarding them. When these equations contain no powers higher than the first, the remainder of the operation will be the same as one or other of the processes described in the last chapter.

*Some skill* is occasionally required in reducing the data of the questions to an Algebraic form, since that may often be done in various ways.

Sometimes we represent the unknown quantity—not by a single letter, but by a letter with a co-efficient, as in Example 2, or with some other qualification.

At other times it is best to regard as the unknown quantity—not that which is directly sought, but some other from which it may be easily deduced. An instance of this will be seen in Example 4, in which $x$ is made to represent the time instead of the distance sought. The object of doing so is to avoid fractions.

Sometimes we are obliged to use several letters, to represent the different quantities or numbers sought, as in Example 6; but in many instances in which this appears, at first sight, to be necessary, we find, on further examination, that it may be avoided, and that we obtain a more elegant solution by using only one letter, as in Examples 3 and 5.

Most of the questions, producing simple equations, may be resolved by pure Arithmetic, but that is no hindrance to their forming good examples in Algebra. The latter, indeed, is almost always the easier method, the arithmetical process requiring more consideration and more ingenuity.

Example 1. What number is that, the double of which as much exceeds 25 as its half falls short of it?

Let $x$ be the number.

$$\text{Then, } 2x-25=25-\frac{x}{2}.$$
$$\text{Hence } \quad 5x=100,$$
$$\text{and } \quad x=20.$$

Example 2. What number is that, whose half, added to its third part, is 10?

Let $6x$ be the number.

$$\text{Then, } 3x+2x=10.$$
$$\text{Hence } \quad x=2,$$
$$\text{and } \quad 6x=12.$$

Example 3. Divide the number 50 into two such parts that the sum of 8 times the greater and 9 times the smaller, shall be 423.

Let $x$ be the smaller part, and $50-x$ will be the greater.

$$\text{Then } 400-8x+9x=423.$$
$$\therefore x=23,$$
$$\text{and } 50-x=27.$$

Example 4. A traveller sets out from Milton to Newton, 35 miles distant, walking at the rate of 4 miles per hour. Another, at the same time, sets out from Newton to Milton at the rate of 3 miles per hour. In what part of the road do they meet?

Let the number of hours they walk before meeting each other be represented by the letter $x$: then the first will have walked $4x$ miles, and the second, $3x$. But these two distances make up the whole distance between the two places.

$$\therefore 4x+3x=35.$$

$$\text{Hence} \quad x=5,$$

$$4x=20,$$

$$\text{and} \quad 3x=15.$$

That is, they meet 20 miles from Milton and 15 from Newton.

**Example** 5. To divide the number 90 into four such parts that, if the first be increased by 5, the second decreased by 4, the third multiplied by 3, and the fourth divided by 2, the results, in all the cases, shall be exactly the same.

Let $x$ represent the result in each case. Then $x-5$ is the first part; $x+4$, the second; $\frac{x}{3}$ the third; and $2x$ the fourth.

$$\text{But } (x-5)+(x+4)+\frac{x}{3}+2x=90.$$

Hence $13x=273$, and $x=21$; and the four parts are 16, 25, 7, and 42.

**Example** 6. What is that fraction which would just be equal to $\frac{1}{2}$ if the numerator were increased by 3; but only to $\frac{1}{4}$ if the numerator were diminished by 2?

Let $x$ and $y$ be the numerator and denominator, and consequently, $\frac{x}{y}$ the fraction.

$$\text{Then } \frac{x+3}{y}=\frac{1}{2}, \text{ and } \frac{x-2}{y}=\frac{1}{4}$$

From these two equations we find $x=7$, $y=20$, and $\frac{x}{y}=\frac{7}{20}$.

**Exercise** 1. What is that number which, with 17 added to it, will make 55? Ans. 38.

2. What number is that whose half exceeds its third part by 17? Ans. 102.

A. Determine a number which, when 365 is taken from it, shall leave 214.

B. What is the number whose third part and fourth part added together make 91?

3. What number is that which is as much short of 100 as its double exceeds 100? Ans. 66$\frac{2}{3}$.

C. A farmer had three times as many cows as horses, and ten times as many sheep as cows. The whole number of his live stock (horses, cows, and sheep) was 306. How many horses had he?

4. A boy being asked how many marbles he had, said he would answer no such question, but that it was a matter of indifference to him whether he gained 15 more, or doubled what he had and then lost 10. Can you compute, from that hint, how many he actually had? Ans. 25.

D. Two tradesmen began business with equal sums of money. The first gained £142, and the second lost £150. The first was then twice as rich as the second. What sum had each at the commencement?

5. Divide 1000 into two such parts, that 9 times the larger shall exceed 13 times the smaller by 970.
Ans. 635 and 365.

6. The sum of two numbers is 154, and their difference 42. What are they?
Ans. 98 and 56.

E. Divide 100 into two such parts that 25 times the less may exceed 24 times the greater by 1.

F. Divide 100 into two such parts, that half the one added to one-seventh of the other shall just make 20.

7. An election, at which there were only two candidates, was gained by Mr Smith, who had a majority of 75 votes over Mr Jones: the number of voters in the interest of Jones was just two-thirds of the number who supported Smith. What was the whole number of voters?
Ans. 375.

G. A charitable gentleman distributed the contents of his purse, which contained 23 shillings, among a poor family, consisting of a man, his wife, and two children. He gave the man twice as much as his wife, and to the oldest child one-third of what he gave to his father. There was then a shilling over, which he gave to the youngest child: How much did each receive?

8. In the House of Commons there are 654 members, Scotland sending half as many as Ireland and half a member more, and England contributing 29 short of 5 times

the number of Irish members. What are the respective numbers? Ans. 496, 105, and 53.

H. Four persons engaged in a speculation requiring an outlay of £2400. Of that sum, *A* contributed twice as much as *B*, and £10 more; *C* contributed £20 less than *A*; and *D* only two-thirds of *C*. What sums did they separately contribute?

9. Two boys, who made their living by selling nuts, commenced the week with the same sum; but, when they met on the Saturday night, the one found that, after paying for his maintenance, he had gained a half-crown; and the other that he had lost 1s. 6d. The consequence was, that the former had now three times as much as the latter. What did they begin the week with? Ans. 3s. 6d. each.

I. Two fields were purchased at £43 and £28 per acre respectively. Their united area was 12 acres, and their united price £411. What were the sizes of the fields separately?

10. A gentleman, after travelling 12 hours without stopping, found that, if he had travelled 3 miles an hour faster, he would have accomplished the journey in two hours less of time. At what rate did he travel?

Ans. 15 miles per hour.

J. Two friends, living at Walton and Middleton, 24 miles apart, agreed to meet on an angling excursion between the two places. The one from Walton set out an hour after the other, but, having no encumbrance, got on at the rate of 4 miles an hour,—while his friend, having undertaken to bring fishing-tackle and provisions for both, could only proceed at three-fourths of that speed. At what point on the road did they meet?

K. A man departs on a journey, walking at the uniform rate of 3 miles an hour; and, two hours later, another sets out after him, riding, at 7 miles an hour. At what distance on the road will the latter overtake the former?

11. If you divide a certain number by 9, and add together divisor, dividend, and quotient, their sum shall be 59. The number is required. Ans. 45.

L. Divide the number 100 into four such parts that the first may be equal to half the second, but greater than the third by 6, and less than the fourth by 1.

12. Find three numbers, such that their sums, taken two by two, shall be 11, 12, and 13. Ans. 5, 6, and 7.

*M. A man and* his wife were married at the respective ages of 40 and 20 years. How old will the man be when his wife's age becomes three-fourths of his own ?

13. A certain reading-room is maintained at the joint expense of the subscribers. At the end of the first year they find they have each £1, 4s. to pay; but one of them remarked correctly that, if there had been seven more of them, the expenditure would not have been increased, and they would have got off at the rate of a sovereign a piece. Compute the number of members. Ans. 35.

N. Half the difference of two numbers added to half the sum is 93·5, and half the difference taken from half the sum is 77·4. What are the two numbers ?

14. What are the two numbers, whose sum is 133, and quotient 18 ? Ans. 126 and 7.

O. Find two numbers whose difference shall be 1, and the difference of their squares 19.

P. Divide 100 into two such parts that the difference of their squares may be 200.

Q. The difference of two numbers is 40, and the difference of their square roots 2. What are the numbers ?

15. A lady, on her birth-day, presented all the wives and children of her cottagers with donations of money. She gave the women a sovereign each, and the children a half-crown. Having done so, she found that she had bestowed 200 donations, and expended £58, 5s. What were the respective numbers of women and children ?

Ans. 38 and 162.

R. Two servants lived together 40 years in the service of the same master, and at the same wages,—viz., £12 per annum. The one spent 10 shillings less every year than the other, and found at the end of the 40 years, that she had saved twice as much as her fellow-servant, and half-a-year's wages besides. What did they severally save in the year ?

16. A broker bought two cabinets at a sale, suspecting that a £10 note was hid in one of the two. If in the one, he thought, it should be worth twice the other; but if in the other, it ought to be three times the value of the first. What values did he put upon the cabinets without the note ? Ans. £6 and £8.

S. Two foolish young gentlemen began to play against each other with equal sums of money. After playing 12 hours, and incurring 15 shillings each of expenses, the one remained with twice as much money as the other, having gained from him £15. What had each when they began ?

17. A lady's ring cost 7 guineas; her brooch cost as much as her ring and half her bracelets, and the bracelets cost as much as the ring and brooch together. What was the cost of the whole? Ans. 56 guineas.

T. On examining a meteorological table for a particular locality, it was found that the whole fall of rain for the year was 22·8 inches,—that the quantity which fell in the first and last quarters together was just equal to that which fell in the second and third,—that the fall in the first quarter was two-thirds of that in the fourth,—and that the third exceeded the second by 2 inches. What were the depths of fall in the respective quarters?

18. A tea-dealer considers that 2 lb. of his best green tea is worth three of his best black. He mixes them together in equal proportions, and sells the mixture at 6s. 3d. per pound. At what rate would he sell the black and green separately? Ans. At 5s. and 7s. 6d.

U. A lady, going to purchase a carpet, found, that if she took the best quality in the shop, at 8 shillings the yard, it would take £1 more than the money she brought with her. She therefore contented herself with the second, at 7s. 6d., and found that she had then £2 left. What did the carpet cost?

19. There are three numbers such, that the first, with half the sum of the other two, makes 46; the second, with one-third of the other two, makes 30; and the third, with one-fourth of the first and second, amounts to 29. Calculate the numbers. Ans. 30, 14, and 18.

V. A boy bought 90 apples and pears for 2s. 4d., having got 3 apples for a penny, and 7 pears for twopence. How many had he of each kind?

20. Two cousins, John and George, go to school at the beginning of the year with 5 guineas a-piece of pocket-money. At the end of the year John finds that he has spent twice as much as George, and 5 shillings more, and that he has remaining 5 shillings less than half of what George still possesses. How much does each return home with? Ans. £1, 10s. and £3, 10s.

W. Mr Johnson, Mr Thomson, and Mr Wilson, being one day in company, were requested to contribute to a charitable purpose. Mr Thomson said, that if Mr Johnson subscribed more than £50, he would give, for his subscription, three times as much as all that Mr Johnson's exceeded that sum; while Mr Wilson offered to table four *times as much* as Mr Thomson's exceeded £50. On hearing this, Mr Johnson, resolving to strain the liberality of his friends, went to the utmost extent of his ability; and Mrs Ellison, who requested their contributions, had the pleasure of departing with £730. What was Mr Johnson's subscription?

21. A gentleman left £8400 to be divided among his four nephews in certain portions named, and directed his house to be sold by auction. As the four nephews were returning together from the sale, Edward observed, that if his uncle had left him twice as much as he did, he could have bought the house with the sum. "He left me enough," said Richard, "to buy it twice over if I were disposed." "It would have taken Harry's and mine put together to purchase it," replied Alfred. "I could have paid for it with my own share," exclaimed Harry, "if Dick had been generous enough to add to it the third part of his." What was the house sold for?

Ans. £2400.

X. A *young* sportsman returning unsuccessful from a day's shooting, and meeting his friend with six brace of black game, offered him his gun, shot-belt, and powder-flask for 5 brace; or the gun and belt for 4 brace; or the belt and powder-flask for 3 brace; or the belt, flask, and 6 guineas for the whole. What values did *he* put upon the different articles and the game?

22. A son, asking his father how old he was, received the following reply:—"Seven years ago I was four times as old as you; but seven years hence, if you and I live, my age will then be only double yours." It is required, from this information, to satisfy the son's curiosity.

Ans. The father's age was 35.

Y. On making up the roll of an army after a battle, it was found that the number of effective men was only 714 more than half the number before the battle. Of the remainder, the wounded were twice as many as the slain, and the prisoners equal to one-third of all that were left for immediate service, while the number of wounded exceeded the number of prisoners by 677. What was the original strength of the army?

23. An itinerant orange-vender bought a quantity of oranges for sale, at the rate of 5 for twopence. He then arranged the good and the bad in two separate baskets, containing equal numbers, and sold the one basketful at 3 a penny and the other at 2 a penny. In selling them he met another orange-vender, who laughed at his simplicity,

and said he would have no profit upon them; but, when he had sold the whole, he found he had gained sixpence. Please to calculate, from these data, how many oranges he bought and sold. Ans. 30 dozen.

Z. Two brothers went from their house to the neighbouring town, taking different roads. The two roads were of equal lengths, but the one was over level ground and the other over a hill,—so that the elder brother, who travelled along the level road, went at the uniform rate of 3 miles an hour; while the younger could advance at only two-thirds of that rate for the first half of the way, thinking to make up for it by doubling his speed the other half. It turned out, however, although he did double his speed, that he was ten minutes longer in reaching his destination. What was the distance of the house from the town?

# CHAPTER XV.

## PROPORTION.

### DEFINITIONS.

1. When the first of four quantities contains the second as often as the third contains the fourth, the four quantities are said to be *Proportional*. Thus, since $\frac{16}{4} \doteq \frac{12}{3}$, we say, 16 : 4 : : 12 : 3; and if $\frac{a}{b} \doteq \frac{c}{d}$, then $a : b :: c : d$.

NOTE. When we speak of the first containing the second as often as the third contains the fourth, we mean that the quotients are equal, whether they be integers, fractions, or mixed numbers, or even if they be roots or other expressions not convertible at all into definite numbers. Thus, 3 : 4 : : 9 : 12, because $\frac{3}{4} \doteq \frac{9}{12}$. Also $2 : \sqrt{20} :: 3 : \sqrt{45}$, although we cannot express either $2 \div \sqrt{20}$ or $3 \div \sqrt{45}$ exactly, in numbers, at all.

2. When four quantities are proportional, the first is said to have to the second the same *Ratio* which the third has to the fourth. A proportion, therefore, expressing *the equality* of two ratios, is sometimes written thus,

16 : 4 = 12 : 3, which is read thus,—the ratio of 16 to 4 is equal to the ratio of 12 to 3.

3. The first term of a ratio is called its *Antecedent*, and the second its *Consequent*. Thus, in the proportion 16 : 4 :: 12 : 3, the antecedents are 16 and 12, and the consequents 4 and 3.

4. The first and fourth terms of a proportion are called the *Extremes;* and the second and third, the *Means*.

5. The fourth term is said to be a *Fourth Proportional* to the other three.

6. When the first of three given quantities has the same ratio to the second, which the second has to the third, the last is called a *Third Proportional*, and the second a *Mean Proportional* to the other two.

### THEOREM I.

*In any Proportion the Product of the Extreme Terms is equal to the Product of the Means.*

Thus, since 3 : 4 :: 9 : 12, we have $3 \times 12 = 4 \times 9$; and, generally, if $a : b :: c : d$, then $ad = bc$.

DEM. If $a : b :: c : d$, then, by Def. 1, $\frac{a}{b} = \frac{c}{d}$. Multiplying both sides of the equation by $bd$, we have $ad = bc$.

NOTE. This theorem forms a good criterion for determining if four given quantities are proportional.

EXERCISE A. Are the numbers 51, 85, 39, and 65 proportional?

B. Is it correct to say,—As 28 : 42 :: 46 : 70?

C. Are the four quantities $8a$, $12b\sqrt{x}$, $10\frac{a}{b}$, and $15\sqrt{x}$ in proportion?

### THEOREM II.

*If three Quantities are proportional, the Product of the Extremes is equal to the Square of the Mean.*

DEM. If $a : b :: b : c$, then, $\frac{a}{b} = \frac{b}{c}$, and $\therefore ac = b^2$.

EXERCISE A. Are the three numbers, 336, 420, and 525, proportionals?

B. Is $36a^2x^2$ a mean proportional between $54ax^3$ and $24a^3x$?

### THEOREM III.

*If the Product of two Quantities be equal to the Product of two others, the four are Proportionals, making the Factors of the one Product the Means, and the Factors of the other the Extremes.*

**DEM.** Since $ad \doteq bc$, dividing both sides of the equation by $bd$, we have $\frac{a}{b} = \frac{c}{d}$, or $\frac{c}{d} = \frac{a}{b}$, and, consequently, by Def. 1, $a : b :: c : d$, or $c : d :: a : b$.

If we divide both sides of the first equation by $ac$, we have $\frac{d}{c} = \frac{b}{a}$, or $\frac{b}{a} = \frac{d}{c}$, and $d : c :: b : a$, or $b : a :: d : c$.

**EXERCISE.** Since $8 \times 15 \doteq 10 \times 12$, show how many proportions can be stated among these four numbers.

### THEOREM IV.

*If four Quantities are Proportionals, they are also Proportionals when stated inversely.*

Thus, if $a : b :: c : d$, then $b : a :: d : c$.

**DEM.** Since $a : b :: c : d$, then, by Th. I., $bc \doteq ad$. Dividing both sides by $ac$, $\frac{b}{a} \doteq \frac{d}{c}$. Therefore $b : a :: d : c$.

**EXERCISE.** Take the proportion, $75 : 100 :: 57 : 76$, and state it inversely.

### THEOREM V.

*If four Quantities are Proportionals, they are also Proportionals when stated alternately.*

Thus, if $a : b :. c : d$, then $a : c :: b : d$.

**DEM.** If $a : b :: c : d$, then, by Th. I., $ad \doteq bc$. Dividing both sides by $cd$, we have $\frac{a}{c} = \frac{b}{d}$ $\therefore a : c :: b : d$.

**Exercise.** Since $20ax : 15bx :: 12ay : 9by$, state the proportion alternately.

## THEOREM VI.

*When four Quantities are Proportionals, we may multiply or divide either of the Extremes by any Number, provided that we multiply or divide one of the Means by the same Number, and the Proportion will still hold good.*

Thus, if $a : b :: c : d$, then $3a : 3b :: c : d$, and $5a : b :: 5c : d$. Also, $\frac{a}{2} : b :: \frac{c}{2} : d$, &c., &c. In general $ma : mb :: nc : nd$, and $ma : nb :: mc : nd$, $m$ and $n$ being any numbers, integral or fractional.

**Dem.** Since $a : b :: c : d$, then $ad = bc$. Multiplying both sides of the equations by $m$, we have $adm = bcm$. That is, $a \times md = b \times mc$, or $ma \times d = mb \times c$, giving by Theorem III., the two propositions, $a : b :: mc : md$, and $ma : mb :: c : d$.

We may also express our equation thus, $ma \times d = mc \times b$, or $a \times md = c \times mb$, giving two proportions thus, $ma : b :: mc : d$, and $a : mb :: c : md$.

Again, returning to our original equation, $ad = bc$, and multiplying both sides by $mn$, we have $admn = bcmn$; that is, $ma \times nd = mb \times nc$, or $ma \times nd = nb \times mc$. $\therefore ma : mb :: nc : nd$, or $ma : nb :: mc : nd$.

If we had used $\frac{1}{m}$ and $\frac{1}{n}$ throughout the preceding steps, instead of $m$ and $n$, we should have found, in like manner, that $a : b :: \frac{c}{m} : \frac{d}{m}$, that $\frac{a}{m} : \frac{b}{m} :: c : d$, that $\frac{a}{m} : b :: \frac{c}{m} : d$, that $a : \frac{b}{m} :: c : \frac{d}{m}$, that $\frac{a}{m} : \frac{b}{m} :: \frac{c}{n} : \frac{d}{n}$, and that $\frac{a}{m} : \frac{b}{n} :: \frac{c}{m} : \frac{d}{n}$.

**Exercise A.** Change the first and second terms of the proportion $40 : 60 :: x : y$ into smaller numbers.

B. Change the proportion $\frac{p}{2} : x :: \frac{q}{2} : y$, into another containing only integral terms.

1. The same with the proportion $\frac{a}{3} : \frac{b}{2} :: \frac{c}{6} : \frac{d}{5}$.

Ans. $2a : 5b :: c : 2d$.

2. Change the proportion $(a+x)^2 : a^2-x^2 :: 15m^2 : 25mn$ into one with simpler terms.

Ans. $a+x : a-x :: 3m : 5n$.

### THEOREM VII.

*If four Quantities be Proportionals, like Powers or Roots of those Quantities will also be Proportionals.*

Thus, if $a : b :: c : d$, then $a^3 : b^3 :: c^3 : d^3$, and $\sqrt{a} : \sqrt{b} :: \sqrt{c} : \sqrt{d}$, &c., &c.

DEM. Since $a : b :: c : d$, it follows, by Th. I., that $ad \doteq bc$. Raising both members of the equation to any power, say the $n$th power, we have $a^n d^n = b^n c^n$. Hence, by Th. III., $a^n : b^n :: c^n : d^n$.

If, instead of raising both members of the equation to the $n$th power, we had extracted the $n$th root, we should have had $\sqrt[n]{a}\sqrt[n]{d}=\sqrt[n]{b}\sqrt[n]{c}$, and $\sqrt[n]{a} : \sqrt[n]{b} :: \sqrt[n]{c} : \sqrt[n]{d}$.

EXERCISE A. Square all the terms of the proportion $25 : 35 :: 15 : 21$, and try if the results are proportional, by Theorem I.

B. Change the proportion $9a^2 : 25b^2 :: 4c^4 : 49d^4$, into one composed of simpler terms.

### PROBLEM I.

*To find any Term of a Proportion, the other three Terms being given.*

RULE. If the required term is one of the means, divide the product of the two extremes by the other mean. But if the required term is one of the extremes, divide the product of the two means by the other extreme.

DEM. If $a : b :: c : d$, then $ad \doteq bc$. Consequently, according as we divide both sides by $d$, $c$, $b$, or $a$, we have $a=\frac{bc}{d}$, $b=\frac{ad}{c}$, $c=\frac{ad}{b}$, or $d=\frac{bc}{a}$.

**Exercises.** Supply the blank term in each of the following proportions :—

(1.) $102 : \text{——} :: 78 : 130.$ Ans. $170.$
(2.) $\text{——} : 84 :: 70 : 98.$ Ans. $60.$
(3.) $13a : 15a :: 13x : \text{——}.$ Ans. $15x.$
(4.) $2a : 3\sqrt{b} :: \text{——} : 21a.$ Ans. $14.\frac{a^2}{\sqrt{b}}.$
(5.) $a+x : \text{——} :: a : b.$ Ans. $b+\frac{b}{a}x.$
(A.) $4 : 5 :: 12 : \text{——}.$
(B.) $20 : 15 :: \text{——} : 39.$
(C.) $14a^2bx : \text{——} :: 21ax^2y : 27bxy^2.$
(D.) $3a^2+3ax : 3a^2-3ax :: \text{——} : 4ax-4x^2.$
(E.) $\text{——} : \frac{x}{y} :: \frac{y}{x} : \frac{b}{a}.$
(F.) $4a^2-9b^2 : 6a^2-5ab-6b^2 :: 6a^2+5ab-6b^2 : \text{——}.$

**Exercise 6.** Two quantities are to each other as 3 to 4. If we call the former of the two $3x$, what must we call the latter; and if we call the latter $60x$, what must we call the former? Answers: $4x$ and $45x$.

G. Two quantities are to each other as 5 to 6. If the former be $150x$, what will be the latter? if the former be $\frac{5}{6}y$, what will be the latter? and if the latter be represented by $112z$, what expression will represent the former?

**Example.** If *A*, working alone, can do a piece of work in 3 days, and *B*, working alone, can do it in 4 days, in what time will they do it working together?

Let $w$ represent the work. Then, in one day,

$$A \text{ does } \tfrac{1}{3}\,w = \tfrac{4}{12}\,w,$$
$$B \text{ does } \tfrac{1}{4}\,w = \tfrac{3}{12}\,w.$$

Consequently, working together, they will do $\frac{7}{12}\,w$ in one day. But the quantity of work done is proportional to the time employed upon it, when all other circumstances are the same.

Therefore $\frac{7}{12}\,w : w :: 1 \text{ day} : \text{time required}.$

The time required is therefore the blank term, which, found by the rule, is $\frac{12}{7}$ of a day, or $1\frac{5}{7}$.

**Exercise 7.** If a gardener alone could dig a piece of ground in 10 days, and his apprentice alone could dig it in 12 days, in what time will both finish it working together? Ans. In $5\frac{5}{11}$ days.

8. There are three pipes which communicate with the same cistern. The first, when running alone, takes 24 hours to fill the cistern; the second, 20 hours; and the third, 18 hours. In what time will they fill it all running together? Ans. In $6\frac{42}{53}$ hours.

H. *A*, working alone, can do a piece of work in 28 days, and *B*, working alone, can do it in 21 days. How long will it take them working together?

K. Three lodgers, *A*, *B*, and *C*, agree to mess in common. How long may they count upon a pound of tea serving them, if it lasted *A* alone 75 days; *B*, 80 days; and *C*, 90?

### Problem II.

*To find a mean Proportional between two given Quantities.*

Rule. Multiply the two given quantities together, and extract the square root of their product.

Dem. If $a : b :: b : c$, $ac \doteq b^2$, by Th. II. $\therefore b \doteq \sqrt{(ac)}$.

Example. Find a mean proportional between $4a^2$ and $9b^2$.

$$\sqrt{(4a^2 \times 9b^2)} \doteq \sqrt{36a^2b^2} = 6ab.$$

Exercise 1. Find a mean proportional between 49 and 144. Ans. 84.

2. Between $9xy^3$ and $64x^3y$. Ans. $24x^2y^2$.

3. Between $(a+x)^2$ and $(a-x)^2$. Ans. $a^2-x^2$.

A. Find a mean proportional between 121 and 64.

B. Between $81a^2$ and $25b^2$.

C. Between $4a^2+12ab+9b^2$ and $9a^2+12ab+4b^2$.

# CHAPTER XVI.

## IRRATIONAL QUANTITIES, OR SURDS.

### DEFINITIONS.

Any root which cannot be expressed without the use of *a radical* sign, or of a fractional exponent, is called an

*Irrational Quantity* or a *Surd*, as $\sqrt{a}$, $\sqrt[4]{(x-y)}$, $\sqrt[3]{(a)^2}$, or $(a+b)^{\frac{2}{3}}$.*

Any quantity expressed without a radical sign, or fractional exponent, is said to be *Rational.*

A rational quantity may be expressed under an irrational, or surd, form, as $\sqrt{(a^4)}$.

Surds are said to be *of the same kind*, when the root is the same in both, as $\sqrt{x}$ and $\sqrt{y}$, or $\sqrt[4]{(x^3)}$ and $\sqrt[4]{(ax^2)}$.

NOTE. If a surd is to be raised to a power the same as that indicated by the exponent of the root, that is done merely by removing the radical sign. Thus the square of $\sqrt{x}$ is $x$, and the fourth power of $\sqrt[4]{(a+b)}$ is $a+b$. This is evident from Definitions 19 and 20 of Ch. I.; for the $n$th root of any given quantity is such that, if $n$ factors, severally identical with it, be multiplied together, they produce the given quantity.

EXERCISE A. What is the square of $\sqrt{(a^6-b^6)}$?
B. Raise $\sqrt[3]{(x^2-y^2)}$ to the third power.
1. Square $5\sqrt{(3ab)}$.† Ans. $75ab$.
C. Square $5\sqrt{x}$, and $2\sqrt{(7z)}$.
D. Raise $6\sqrt[3]{c}$ and $2\sqrt[3]{(3d)}$ each to the third power.
2. Multiply $3\sqrt{(5y+11)}$ by $7\sqrt{(5y+11)}$.
Ans. $105y+231$.
E. What is the product of $14\sqrt{(2x-3y)}$ and $13\sqrt{(2x-3y)}$?
F. What is the product of $3\sqrt[3]{x}$, $5\sqrt[3]{x}$, and $4\sqrt[3]{x}$?
3. Multiply $2x-3\sqrt{y}$ by $3x+4\sqrt{y}$.
Ans. $6x^2-x\sqrt{y}-12y$.
G. Square $1+\sqrt{x}$, and $x-\sqrt{(ay)}$ separately.
H. Multiply $3+2\sqrt{a}$ by $3-2\sqrt{a}$.
K. Multiply $7a+5\sqrt{(bc)}$ by $3a-2\sqrt{(bc)}$.

## PROBLEM I.

### *To reduce a Quantity in the Form of a Surd to its simplest Form.*

RULE. Extract the root of every factor which admits of it and prefix the radical sign to the remaining factors.

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* Fractional exponents are rarely used in this volume. They may be introduced more appropriately at a subsequent stage.

† No other direction is required for this and the following Exercise than the remark contained in the Note above, and the rules given in Chapter v.

Thus $\sqrt{(a^2b)} \doteq a\sqrt{b}$, $\sqrt[4]{(a^8)} \doteq a^2$, $\sqrt[3]{(8x^3y^2)} \doteq 2x\sqrt[3]{(y^2)}$, and $\sqrt{(50a^2b^3)} = \sqrt{(2 \times 25 \times a^2 \times b^2 \times b)} \doteq 5ab\sqrt{(2b)}$.

DEM. Since a quantity made up of several factors is raised to any power by raising the factors separately to that power, it follows that any root of a quantity made up of several factors is found by extracting the roots of the factors separately, since evolution is merely a reversal of the process of involution. Those factors which have rational roots may have them expressed without the radical sign, but the remainder must have their roots expressed by means of that sign, or, which is the same thing, by the use of fractional exponents.

NOTE 1. This, so far, is little more than a repetition of Pr. I. of Ch. VIII.

EXERCISE A. Express in their simplest form, $\sqrt{a^6}$, and $\sqrt[3]{(b^3c)}$.

B. The same with $\sqrt[3]{(64x^3y)}$ and $\sqrt{(16a^4x)}$.

C. The same with $\sqrt{(144ab^2c^4)}$ and $\sqrt[3]{(125x^3y^6z^2)}$.

D. The same with $\sqrt{(300a^2x)}$ and $\sqrt[3]{(54x^3y^4)}$.

NOTE 2. If any of the factors are fractional, or if the surd is itself in the form of a fraction, the same method is pursued (and for a similar reason), the denominators, or factors of the denominator, being treated as other factors. Thus $\sqrt{\frac{a}{9}} \doteq \frac{\sqrt{a}}{3}$ and $\sqrt[3]{\frac{1}{8}x} \doteq \frac{1}{2}\sqrt[3]{x}$.

EXERCISE E. Express in a simpler form $\sqrt{\left(\frac{1}{4}y\right)}$ and $\sqrt{\frac{y}{9}}$.

F. The same with $\sqrt[3]{\left(\frac{1}{125}x^3y^2\right)}$, and $\sqrt[3]{\frac{27a}{8b^3}}$.

## PROBLEM II.

*To multiply Surds, of the same Kind, together.*

RULE. Multiply together the quantities affected by the common radical sign, and prefix that sign to the product.

DEM. Since $(ab)^2 \doteq a^2b^2$, it follows that $\sqrt{(a^2b^2)} \doteq ab = a \times b = \sqrt{(a^2)} \times \sqrt{(b^2)}$. $\therefore \sqrt{(a^2)} \times \sqrt{(b^2)} \doteq \sqrt{(a^2b^2)}$. Now, since $a$ and $b$ may be any quantities whatever, so *also may* $a^2$ and $b^2$. Writing therefore $c$ and $d$ for $a^2$ and

$b^2$, we have $\sqrt{c}\times\sqrt{d}=\sqrt{(cd)}$. The same proof applies equally to other roots than the square root, so that, in general, $\sqrt[n]{c}\times\sqrt[n]{d}=\sqrt[n]{(cd)}$.

If there are more quantities than two to be multiplied together, the demonstration is the same.

Exercise 1. Multiply $\sqrt[3]{(7y)}$ by $\sqrt[3]{(6z)}$. Ans. $\sqrt[3]{(42yz)}$.

A. Multiply $\sqrt[3]{x}$ by $\sqrt[3]{y}$, and $\sqrt{(21a)}$ by $\sqrt{(2b)}$.

B. Multiply $\sqrt{(x+3)}$ by $\sqrt{(x-3)}$.

C. What is the product of $+\sqrt[3]{(2a)}$, $-\sqrt[3]{(3b)}$, and $+\sqrt[3]{(x^2)}$?

Note 1. After multiplying, the product may sometimes be brought into a simpler form, by Pr. I.

Exercise 2. Multiply $\sqrt{(2ab)}$ by $\sqrt{(6ac)}$.

Ans. $2a\sqrt{(3bc)}$.

D. Multiply $\sqrt{(15xy)}$ by $\sqrt{(21yz)}$, and $\sqrt[3]{(4x^2)}$ by $\sqrt[3]{(2x)}$.

E. Simplify the expression, $\sqrt{(2abc)}\times\sqrt{(3acd)}\times\sqrt{(6bd)}$.

F. The same with $\sqrt[3]{(50a^2xy)}\times\sqrt[3]{(2ax^2y)}\times\sqrt[3]{(10a^2y^2)}$.

Note 2. If any of the quantities to be multiplied together are rational quantities, or have rational factors, multiply the rational factors, or quantities, by themselves, and the irrational by themselves; and the product of these two will be the product of the whole.

Exercise 3. Multiply $3\sqrt{(2x)}$ by $4\sqrt{(3y)}$.

Ans. $12\sqrt{(6xy)}$.

4. Multiply $10\sqrt[3]{(5x)}$ by $2\sqrt[3]{(25x^2)}$. Ans. $100x$.

G. Multiply $5\sqrt{(ab)}$ by $6\sqrt{(ac)}$.

H. Multiply $2\sqrt{(3ax)}$ by $7\sqrt{(3ay)}$.

K. Square $4\sqrt{x}+5\sqrt{y}$.

L. Multiply $2\sqrt{x}+5$ by $\sqrt{x}-2$.

M. What is the product of $+2\sqrt[3]{(abc)}$, $-3\sqrt[3]{(a^2c)}$, and $-4\sqrt[3]{(b^2c)}$?

Note 3. The rule is equally applicable if any of the quantities, or of the factors, are not actual surds, but merely under that form. In this case the result must be reduced to its simplest form after multiplying: or we may, if we please, reduce the quantities to their simpler forms previous to multiplication.

Exercise N. Multiply $\sqrt{(4xy)}$ by $\sqrt{(9xz)}$.

O. Multiply $\sqrt[3]{(3x^5)}$ by $\sqrt[3]{(8x)}$.

### PROBLEM III.

*To divide one Surd by another of the same Kind.*

**RULE.** Find the quotient of the quantities affected by the common radical sign, and prefix that sign.

**DEM.** We have proved that $\sqrt[n]{c} \times \sqrt[n]{d} \doteq \sqrt[n]{(cd)}$.

$$\therefore \sqrt[n]{(cd)} \div \sqrt[n]{c} \doteq \sqrt[n]{d} = \sqrt[n]{(cd \div c)}.$$

**EXERCISE** 1. Divide $\sqrt[3]{(42yz)}$ by $\sqrt[3]{(6z)}$. Ans. $\sqrt[3]{(7y)}$.

A. Divide $\sqrt[4]{(ab)}$ by $\sqrt[4]{a}$, and $\sqrt{(30xy)}$ by $\sqrt{(3y)}$.

**NOTE** 1. If there are rational factors along with the surds, divide the rational factors by the rational, and the surd by the surd. The product of the two quotients will be that which we require.

**EXERCISE** B. Divide $20\sqrt{15}$ by $4\sqrt{5}$.

C. Divide $45\sqrt[3]{21}$ by 9, and $45\sqrt[3]{21}$ by $\sqrt[3]{7}$.

D. Divide $12\sqrt{(6xy)}$ by $4\sqrt{(3y)}$, and $2\sqrt[3]{(x^4y)}$ by $3\sqrt[3]{(xy)}$.

2. Divide $6 + \sqrt{x} - 2x$ by $2 - \sqrt{x}$. Ans. $3 + 2\sqrt{x}$.

3. Divide $6y - 2\sqrt{y} - 20$ by $3\sqrt{y} + 5$. Ans. $2\sqrt{y} - 4$.

E. Divide $a - 25$ by $\sqrt{a} + 5$.

F. Divide $9 + 6\sqrt{a} + a$ by $3 + \sqrt{a}$.

G. Extract the square root of $225 - 30\sqrt{y} + y$.*

H. Find the square root of $4x - 12\sqrt{(xy)} + 9y$.

**NOTE** 2. Peculiar provisions are required in some cases, into which it is not necessary to enter here.

## CHAPTER XVII.

## SIMPLE EQUATIONS.

(*Continued from Chapter XIII*).

**EXAMPLE** 1. Given $x + 3 : x - 3 :: 5 : 4$, to find the value of $x$.

Any proportion is changed into an equation, by making the product of the extreme terms equal to the product of the means, according to Th. I. of Ch. XV. When this has been done, the value of the unknown quantity or quantities is found by the usual methods of equations.

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* For this and the following Exercise no directions are necessary beyond those of this and the preceding Problem, and of Problem II. of Chapter VIII.

In this example, the product of the extremes is $4x + 12$, and that of the means, $5x - 15$.

$$\therefore 5x - 15 \doteq 4x + 12.$$
$$\text{Hence, } x \doteq 27.$$

**Exercise** 1. Given $2x : x + 20 :: 9 : 10$. Ans. $x = 16\frac{4}{11}$.

A. Given $y - 17 : y + 17 :: 1 : 2$.

2. ... $\frac{10 + x}{5} : \frac{4x - 9}{7} :: 14 : 5$. Ans. $x = 4$.

3. ... $5 : 4 :: \frac{17 - 4x}{4} : \frac{15 + 2x}{3} - 2x$. Ans. $x = 3$.

B. ... $\frac{8x - 3}{5} : \frac{7x + 8}{4} :: 4 : 5$.

C. ... $\frac{x + 5}{3} : \frac{21 - x}{2} :: 2 : 3\frac{1}{2}$.[*]

D. ... $\frac{y - 13}{6} : \frac{y + 13}{12} :: \frac{2}{3} : \frac{3}{4}$.

E. ... $16 : 9 :: \frac{144}{x - 7} : \frac{144}{x + 7}$.

F. ... $\frac{4}{z - 3} : \frac{5}{z - 2} :: a : \frac{3a}{4}$.

G. ... $x^2 - 49 : x :: 2x + 14 : 3$.

H. ... $5x + 3 : 9x - 3 :: 7x - 5 : 8x + 5$.

**Example** 2. Given $5\sqrt[3]{}(3x) = 60$, to find $x$.

Dividing by 5, $\sqrt[3]{}(3x) \doteq 12$.
By Prin. 5, $3x \doteq 1728$.
Hence, $x \doteq 576$.

**Exercise** 4. Given $\sqrt{}(7x) = 35$. Ans. $x = 175$.

5. Given $5\sqrt{x} + 12 = 27$. Ans. $x = 9$.

K. ... $\sqrt{}(45y) = 15$.

L. ... $9\sqrt{}(18z) = 108$.

6. ... $\sqrt{}(y - 35) = 15$. Ans. $y = 260$.

7. ... $\sqrt[3]{}(4z + 3) = 3$. Ans. $z = 6$.

8. ... $\sqrt{}(19x + 13) = \sqrt{}(59 - 4x)$. Ans. $x = 2$.

M. ... $\sqrt{}(3y - 11) = 67$.

N. ... $3\sqrt{}(15x + 60) = \sqrt{}(180x)$.

O. ... $2\sqrt{}(5y + 11) = \sqrt{}(25y - 66)$.

---

[*] In proportions, such as this and several which follow, advantageous use may be made of Theorem VI. of Chapter XV. before forming an equation.

9. Given $\sqrt{(y+9)} = 1 + \sqrt{y}$. Ans. $y = 16$.

P\. ... $\sqrt{(x+5)}\,\sqrt{(x-5)} = x - 1$.

Q\. ... $\sqrt{\frac{y}{4}} + 12 = \sqrt{(4y)}$.

R\. ... $\sqrt{(3x+31)} \times \sqrt{(3x-9)} = 3x + 3$.

S\. ... $\sqrt{(x+5)} : \sqrt{(x-2)} :: \frac{2}{3} : \frac{1}{2}$.*

T\. ... $1 + \sqrt{x} : \sqrt{x} + 7 :: \sqrt{x} - 7 : 1 - \sqrt{x}$.

10. ... $\left.\begin{cases} x+y : x-y :: 3 : 1, \\ 3x+4 : 6y-4 :: 5 : 3. \end{cases}\right\}$ Ans. $\begin{cases} x = 5\frac{1}{3}, \\ y = 2\frac{2}{3}. \end{cases}$

11. ... $\begin{cases} \dfrac{5x+13}{2} - \dfrac{8y-3x-3}{6} = 9 + \dfrac{7x-3y}{3}, \\ \dfrac{x+7}{6} : 2x + \dfrac{3y-8}{8} :: 4 : 21. \end{cases}$

Ans. $x = 5$, $y = 4$.

U\. ... $\begin{cases} 4y - 3x = 12, \\ 3x : 4y :: 5 : 7. \end{cases}$

W\. ... $\begin{cases} \dfrac{11x-y+5}{9} - \dfrac{4(y-2)}{15} = \dfrac{3(x+y+4)}{10} + 28 \\ \dfrac{x+6}{10} : \dfrac{y-2}{15} :: 7\frac{1}{2} : 2\frac{1}{2}. \end{cases}$

# CHAPTER XVIII.

## QUESTIONS PRODUCING SIMPLE EQUATIONS.

(*Continued from Chapter XIV*).

Example. What two parts, in the proportion of 4 to 5, make up a line of 100 feet?

Let the two parts be $4x$ and $5x$.

$$\text{Then,}\quad 4x + 5x = 100$$
$$\text{Hence}\quad x = 11\tfrac{1}{9},$$
$$4x = 44\tfrac{4}{9},$$
$$5x = 55\tfrac{5}{9}.$$

*Or thus*:—Let $x$ be the smaller part: then $100 - x$ will be the greater; and

$$x : 100 - x :: 4 : 5.$$
$$\therefore\ 5x = 400 - 4x.$$
$$\text{Hence, } x = 44\tfrac{4}{9}, \text{ and}$$
$$100 - x = 55\tfrac{5}{9}.$$

---

* As a preparatory step, apply Theorem VII. of Chapter XV.

**Exercise** 1. Two numbers are to each other in the proportion of 4 to 5; but if 5 be taken from the smaller, and 11 added to the greater, the one is then double of the other. What are the numbers? Ans. 28 and 35.

2. Find two numbers such, that, if 6 be added to each, they shall be to each other as 4 to 5; and, if 4 be taken from each, they shall be to each other as 2 to 3.

Ans. 14 and 19.

A. Divide 100 into two such parts, that the excess of the first above 49, shall be to the excess of 49 above the second, as 12 to 11.

B. A school was divided into three classes, the number in the first class being to that in the second as 3 to 4, and the latter to the number in the third, as 7 to 8. There were 243 in all: how many were in each class?

C. Two ships' boats were sent out on a whaling excursion, with crews in the proportion of 3 to 5. Meeting sometime after, they considered it would be better to divide the hands equally between the two boats, and did so by removing 6 men from the one to the other. What was the whole number sent out?

# CHAPTER XIX.

## QUADRATIC EQUATIONS.

### DEFINITIONS.

A *Quadratic Equation*, as has been already defined, is one which contains the second power of the unknown quantity, and no higher.

When the second power occurs alone without the first, the equation is called a *Pure Quadratic*, as $ax^2 = b^3$.

When both the first power and the second enter into it, it is called an *Adfected Quadratic*, as $x^2 + bx = c^2$.

### PROBLEM I.

*To resolve a Pure Quadratic Equation, involving only one unknown Quantity.*

**Rule.** Find the value of the square of the unknown quantity in exactly the same manner as we found the value of the simple quantity in Chapters XIII. and XVII. Then extract the square root of each side according to the sixth General Principle.

NOTE. If the value of the square is a negative quantity, the extraction of its square root is impossible; but, if positive, it may have two square roots, one positive and the other negative. In the resolution of equations, we must attach both signs to the root, since the one answers as well as the other.

When the square of the unknown quantity is found equal to a negative quantity, we can have no real value of the unknown quantity. The statement conveyed in the given equation must have been an impossibility.

EXAMPLE. Given $2x^2 - 45 = 200 - 3x^2$.

Transposing, $5x^2 \doteq 245$,
Dividing by 5, $x^2 \doteq 49$.

Extracting the sq. root of both sides, $x \doteq \pm 7$.

EXERCISE 1. Given $4x^2 - 85 = 90 - 3x^2$. Ans. $x = \pm 5$.

A. Given $9x^2 - 126 = 99$.

2. ... $\frac{x^2}{2} + \frac{x^2}{3} = \frac{x^2}{4} + 21$. Ans. $x = \pm 6$.

B. ... $\frac{7x^2}{12} - 7x + 21 = \frac{7}{12}$.

3. ... $\frac{6x^2 - 4}{3} - 2 = \frac{18 - 4x^2}{3} + x^2$. Ans. $x = \pm 2$.

C. ... $\frac{25x^2 + 147x}{49} - 10x = 100 - 7x$.

D. ... $\frac{24x - 49}{5x^2} + \frac{46}{5x} = 5$.

E. ... $y^2 : 5y - 15 :: 5y - 15 : 16$.

4. ... $\sqrt{(x+1)} \times \sqrt{(x-1)} = x - \frac{1}{x}$. Ans. $x = \pm 1$.

## PROBLEM II.

*To resolve an Adfected Quadratic Equation, involving only one unknown Quantity.*

RULE. Having simplified the equation, when necessary, by removing fractions, and by bringing all the terms containing only known quantities to the right side, and those containing the unknown quantity to the left, collect all the terms containing the square of the unknown quantity into one, and all those containing its first power into another. We shall then have the equation in the form $ax^2 \pm bx = \pm c$.

Next, if the co-efficient ($a$) of the first term is not 1, divide both sides of the equation by that co-efficient. The equation will then appear in the form $x^2 + mx = n$, $m$ and $n$ being any known quantities, positive or negative.

To both sides add the square of half the co-efficient of the second term: the equation will then be of the form $x^2 + mx + \frac{m^2}{4} = \frac{m^2}{4} + n$. This is called *completing the square*, since the left side of the equation is now a complete square.

Extract the square root of each side, and we have then the equation in the form $x + \frac{m}{2} = \surd\left(\frac{m^2}{4} + n\right)$, and consequently, $x = \surd\left(\frac{m^2}{4} + n\right) - \frac{m}{2}$ or $x = \frac{1}{2}(\sqrt{m^2 + 4n} - m)$.

NOTE 1. If $n$ is a negative quantity and greater than $\frac{1}{4}m^2$, then $\surd(m^2 + 4n)$ is an impossible quantity, and no answer can be found to the question. It shows that the conditions of the proposed equation or question are inconsistent with each other. But if $m^2 + 4n$ is a positive quantity, its square root may either be positive or negative; consequently $x$ will have two values, either of which will satisfy the conditions of the given equation.

EXAMPLE 1. Given $x^2 - 8x = 33$, to find $x$.

Completing the square by adding to each side $(\frac{8}{2})^2$, or 16, we have

$$x^2 - 8x + 16 = 49.$$

Extracting the square root of each side,

$$x - 4 = \pm 7.$$

Consequently, $x = 4 \pm 7 = +11$ or $-3$.

EXERCISE 1. Given $x^2 - 4x = 12$. Ans. $x = +6$ or $-2$.
2. Given $x^2 + 10x = 75$. Ans. $x = +5$ or $-15$.
A. ... $x^2 + 12x = 160$.

EXAMPLE 2. Given $7x^2 - 63x + 56 = 0$, to find $x$.

Transposing 56 and dividing by 7,

$$x^2 - 9x = -8.$$

Completing the square,

$$x^2 - 9x + \left(\frac{9}{2}\right)^2 = \frac{81}{4} - 8 - \frac{49}{4}.$$

Extracting the square root,

$$x-\frac{9}{2} \doteq \pm\frac{7}{2}.$$

$$\therefore x \doteq \frac{9}{2} \pm \frac{7}{2} = 8 \text{ or } 1.$$

**Exercise** 3. Given $5x^2-20x+15=0$.
Ans. $x=3$ or $1$.

4. Given $x^2+5x-4=100$. Ans. $x=8$ or $-13$.

5. ... $11x^2-88x+110=209$. Ans. $x=9$ or $-1$.

B. ... $y^2-15y+55=1$.

C. ... $3x^2-18x+25=190$.

6. ... $x^2-10x-100=0$.
Ans. $x=16\cdot18034-$, or $-6\cdot18034-$.

D. ... $y^2-11y+15=0$. To determine the value of $y$ true to four decimal places.

7. Given $\frac{x^2}{12}+\frac{x}{2}=6$. Ans. $x=6$ or $-12$.

8. ... $\frac{1}{6}(z^2-z)=7$. Ans. $z=7$ or $-6$.

9. ... $\frac{3}{2}x^2-x=4$. Ans. $x=2$ or $-\frac{4}{3}$.

E. ... $\frac{z^2-3z+2}{4}=\frac{1}{9}$.

F. ... $\frac{x^2}{9}-\frac{4x}{5}=13$.

G. ... $\frac{x^2}{3}-\frac{2x}{5}=\frac{8}{15}$.

H. ... $\frac{y^2}{4}-\frac{y}{5}=7\cdot8$

10. ... $z^2-8z+20=0$. Ans. $z=4\pm\sqrt{-4}$.

11. ... $\frac{x^2}{3}-\frac{x}{5}-70=2$. Ans. $x=15$ or $-14\frac{2}{5}$.

12. ... $\frac{125-3x}{x}-4x=2$. Ans. $x=5$ or $-6\frac{1}{4}$.

K. ... $\frac{20}{x}-\frac{10x+8}{x^2}-2=0$.

L. ... $\frac{7x+2}{5}-\frac{2x+4}{3}-\frac{12}{x}+\frac{4x-1}{15}=0$.

13. ... $2y-\frac{8y-6}{y+3}=9$. Ans. $y=7$ or $-\frac{2}{3}$.

M. Given $4-x-\frac{4x-11}{2x-3}=\frac{x-1}{3}$.

N. ... $\frac{6x^2-5x}{2}-\frac{3x^3+5x^2-17x-26}{x-1}=-47$.

O. ... $\frac{z-2}{z+2}+\frac{z-4}{z+4}=\frac{119}{78}$.

P. ... $\frac{x^2-7x-3}{x-2}-\frac{7x}{15}+\frac{2x-3}{9}-5=0$.

14. ... $z : z+40 :: 10 : z-40$.
Ans. $z=57{\cdot}0156$ &c., or $-7{\cdot}0156$ &c.

Q. ... $2y+2 : 4y-10 :: 3y-9 : 3y+6$.

15. ... $\sqrt[3]{(z^2-6z)}=-2$. Ans. $z=4$ or $2$.

R. ... $\sqrt{(z+1)}\times\sqrt{(z-4)}=\frac{1}{2}z+2$.

16. ... $x+\sqrt{x}=15$. Ans. $x=\frac{1}{2}(31\pm\sqrt{61})$.

Note 2. When the equation has been reduced to its simplest form, and the terms arranged according to the powers of the unknown quantity, it may be resolved as a quadratic although the highest power may not be the square, provided that the exponent of the power in the first term be double that in the second, and that the unknown quantity do not appear in any other term. In this case, however, we do not at once find the unknown quantity itself, but that power of it which appears in the second term; and from that,—by extracting the root indicated by its exponent, we obtain the first power.

Example 3. Given $z^6-10z^3=459$, to find $z$.

Completing the square, $z^6-10z^3+25 \doteq 484$.

Extracting the sq. root, $z^3-5 \doteq \pm 22$.

$\therefore z^3 \doteq +27$ or $-17$.

Extracting the cube root, $z \doteq +3$, or $-2{\cdot}57$ &c.

Exercise 17. Given $x^4+2x^2=10200$.
Ans. $x=\pm 10$ or $\pm\sqrt{(-102)}$.

18. Given $x^6-35x^3+216=0$. Ans. $x=2$ or $3$.

19. ... $y-\sqrt{y}=20$. Ans. $y=(+5)^2$ or $(-4)^2$.

S. ... $16x^4-12x^2+2=0$.

T. ... $5x^6-120x^3-405=0$.

U. ... $x^8-10x^4=96$.

W. ... $x-12\sqrt{x}-45=0$.

X. ... $y^{\frac{1}{2}}+6y^{\frac{1}{4}}=16$.

## PROBLEM III.

*To resolve a Pure or Adfected Quadratic Equation, involving two or more unknown Quantities.*

RULE. Eliminate all the unknown quantities except one, in the same manner as in simple equations, and then find the value of the remaining unknown quantity by Problem I. or II. of this Chapter.

NOTE. The process will often require to be diversified according to circumstances, affording full scope for the exercise of ingenuity. Sometimes instead of first eliminating one of the unknown quantities, it will be better to form a complete square containing both the unknown quantities, or sometimes even two such complete squares, and then, on extracting the root, simplify one or both equations before the process of elimination commences.

It will frequently happen that equations, which appear to be simple, will assume the quadratic form when one or more unknown quantities are eliminated, and those which appear to be quadratic will rise to a higher order.

EXAMPLE 1. Given $\left\{\begin{array}{l} x^2+y^2=85, \\ x^2-y^2=13, \end{array}\right\}$ to find $x$ and $y$.

Taking successively the sum and difference of the two given equations, we have

$$2x^2=98, \text{ and } 2y^2=72.$$

$$\therefore x^2=49, \text{ and } y^2=36.$$

$$\text{Hence, } x=\pm 7, \text{ and } y=\pm 6.$$

EXAMPLE 2. Given $\left\{\begin{array}{l} x^2-y^2=75, \\ \quad xy\ \,=50, \end{array}\right\}$ to find $x$ and $y$.

Multiplying the first eq. by $x^2$ and squaring the second,

$$x^4-x^2y^2=75x^2,$$

$$\text{and } x^2y^2=2500.$$

Add these two equations together and transpose $75x^2$,

$$x^4-75x^2=2500.$$

Complete the square, and extract the sq. root,

$$x^2-\frac{75}{2}=\pm\frac{125}{2}.$$

$$\text{Hence, } x^2=+100 \text{ or } -25,$$

$$\text{and } x=\pm 10 \text{ or } \pm\sqrt{(-25.)}$$

$$\therefore y^2=25 \text{ or } -100, \text{ and } y=\pm 5 \text{ or } \pm\sqrt{-100}.$$

**EXAMPLE 3.** Given $\left\{\begin{array}{l} x^2+y^2=125, \\ x\;+y\;=15, \end{array}\right\}$ to find $x$ and $y$.

Squaring the second equation,

$$x^2+2xy+y^2\doteq 225.$$

Subtracting this from twice the first,

$$x^2-2xy+y^2\doteq 25.$$

Extracting the square root, $x-y\doteq\pm 5$.

From this equation and the second, by taking their sum and difference, we have

$$2x=20 \text{ or } 10, \text{ and } 2y=10 \text{ or } 20.$$

$$\therefore x\doteq 10 \text{ or } 5, \text{ and } y\doteq 5 \text{ or } 10.$$

**EXERCISE 1.** Given $\left\{\begin{array}{l} 3x^2+4y^2=43, \\ 2x^2-3y^2=\;\;6. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x=\pm 3, \\ y=\pm 2. \end{array}\right.$

A. Given $\left\{\begin{array}{l} 8x^2-5y^2=20, \\ 6x^2-3y^2=42. \end{array}\right.$

2. ... $\left\{\begin{array}{l} x+y : x-y :: 5 : 1, \\ 4 : x :: y : 24. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x=\pm 12, \\ y=\pm\;\;8. \end{array}\right.$

B. ... $\left\{\begin{array}{l} x-y : y :: 1 : 4, \\ xy=180. \end{array}\right.$

3. ... $\left\{\begin{array}{l} x\times z=9, \\ x\div z=4. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x=\pm 6, \\ z=\pm 1\frac{1}{2}. \end{array}\right.$

C. ... $\left\{\begin{array}{l} x^2+y^2=1+\frac{1}{144}, \\ xy=\frac{1}{2}. \end{array}\right.$

4. ... $\left\{\begin{array}{r} x-3y=\;\;9, \\ y^2-4y+2x=26. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x=15 \text{ or } -3, \\ y=\;\;2 \text{ or } -4. \end{array}\right.$

D. ... $\left\{\begin{array}{l} x : x-y :: 22 : 15, \\ x^2-xy=36\frac{2}{3}. \end{array}\right.$

5. ... $\left\{\begin{array}{l} 5x^2-8y^2=+72, \\ 5x\;-8y\;=-12. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x=12 \text{ or } -4, \\ y=\;\;9 \text{ or } -1. \end{array}\right.$

E. ... $\left\{\begin{array}{l} (y-2z)^2 : (y+2z)^2 :: 1 : 25, \\ yz=48. \end{array}\right.$

6. ... $\left\{\begin{array}{l} x+y=34, \\ xy=273. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x=13 \text{ or } 21, \\ y=21 \text{ or } 13. \end{array}\right.$

F. ... $\left\{\begin{array}{l} 5x^2-7y^2=17\frac{4}{9}, \\ 4x^2-6y^2=11\frac{7}{9}. \end{array}\right.$

7. ... $\left\{\begin{array}{r} x^2+y^2=32{\cdot}5, \\ xy=15{\cdot}75. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x=8{\cdot}5 \text{ or } 4{\cdot}5. \\ y=4{\cdot}5 \text{ or } 8{\cdot}5. \end{array}\right.$

G. ... $\left\{\begin{array}{l} x^2+9y^2-6xy=256, \\ x^2-9y^2=352. \end{array}\right.$

8. ... $\left\{\begin{array}{l} x^2+y^2=8\frac{1}{2}, \\ x\;-y\;=1. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x=2\frac{1}{2} \text{ or } -1\frac{1}{2}, \\ y=1\frac{1}{2} \text{ or } -2\frac{1}{2}. \end{array}\right.$

H. ... $\left\{\begin{array}{l} x^2+xy=3276, \\ xy+y^2=5005. \end{array}\right.$

9. ... $\left\{\begin{array}{r} x+y=31{\cdot}25, \\ \sqrt{x}+\sqrt{y}=7{\cdot}5. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x=25 \text{ or } 6\frac{1}{4}, \\ y=6\frac{1}{4} \text{ or } 25. \end{array}\right.$

K. Given $\begin{cases} x+z : x-z :: 11 : 3, \\ (x-5) \times (z-3) = 45. \end{cases}$

10. ... $\left.\begin{cases} x^2-xy = \frac{9}{16}, \\ xy-y^2 = \frac{9}{25}. \end{cases}\right\}$ Ans. $\begin{cases} x = \pm 1\frac{1}{4}, \\ y = \pm \frac{4}{5}. \end{cases}$

L. ... $\begin{cases} x+y = 22, \\ 5xy-5(x-y) = 610. \end{cases}$

11. ... $\left.\begin{cases} x^3+y^3 = 35, \\ x+y = 5. \end{cases}\right\}$ Ans. $\begin{cases} x = 3 \text{ or } 2, \\ y = 2 \text{ or } 3. \end{cases}$

M. ... $\begin{cases} 3x-4z = 17, \\ x^2+z^2 = 274. \end{cases}$

12. ... $\left.\begin{cases} x+y+z = 12, \\ x+2y+3z = 26, \\ x^2+y^2+z^2 = 50. \end{cases}\right\}$ Ans. $\begin{cases} x = 3, \\ y = 4, \\ z = 5. \end{cases}$

N. ... $\begin{cases} y-z = 1, \\ y^2z^2+16yz = 2436. \end{cases}$

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# CHAPTER XX.

## QUESTIONS PRODUCING QUADRATIC EQUATIONS.

NOTE. When the question has been expressed in the form of a quadratic equation, and that equation resolved by the rules in the preceding Chapter, double values of the unknown quantities will be obtained, and each of these values *may* satisfy the conditions of the question (as in Example 1): but it is equally possible that one of the values may not be applicable as an answer to the question, although quite satisfactory as far as regards the equation. This may arise either from the negative form of the value obtained (as in Example 2), or from some other peculiar cause (as in Example 3).

The reason that the equation may be satisfied by the second value while the question is not, arises from the fact of some condition being stated or understood in the question, which has not been expressed, and probably could not be expressed, in the equation.

EXAMPLE 1. What number is that which exceeds the square of its half by $\frac{3}{4}$?

Let $x$ be the number. Then

$$x-\left(\frac{x}{2}\right)^2 = \frac{3}{4}.$$

This equation, resolved as usual, gives $x = 3$ or 1, either of which numbers fulfils the conditions of the question.

**Example 2.** A party of labourers were sent to remove a bank of earth containing 350 cubic yards: but, just as they were commencing, four of them were disabled by an accidental fall of part of the bank: in consequence of this it was found that the rest had each ten yards more to remove. What was the number of the party at first?

Let $x$ express the number at first, and consequently $x-4$ the number after the accident. Then

$$\frac{350}{x} + 10 = \frac{350}{x-4}.$$

If we resolve this by the rules of the preceding Chapters, we find $x = 14$ or $-10$. Either of these numbers would satisfy the equation, but only the first will be applicable to the question, since the supposition of a negative number of men being sent is ridiculous.

**Example 3.** A number, expressed by two digits, is subtracted from its reverse. It is then found that the remainder is three fourths of the original number, but greater than the square of the smaller digit by 20. What is the number?

Let $x$ be the right-hand digit and $y$ the left, of which $y$ must be the smaller, otherwise the remainder would be negative. Then,

$$\begin{array}{rl} 10y + \phantom{0}x \doteq & \text{the number,} \\ 10x + \phantom{0}y \doteq & \text{the reverse,} \\ \hline \text{and } 9x - 9y \doteq & \text{the remainder.} \end{array}$$

$$\text{Hence, } 9x - 9y \doteq \tfrac{3}{4}(10y + x),$$
$$\text{and } 9x - 9y \doteq y^2 + 20.$$

From the former of these two equations we obtain $x = 2y$; and, inserting $2y$ for $x$ in the latter, we find

$$y^2 - 9y = -20;$$

From which we find $y = 5$ or 4, and $\therefore x = 10$ or 8.

But of these two values the second only is applicable, since $y$ cannot be 5 unless $x$ be 10, which cannot be a digit. Therefore the number is 48.

**Exercise 1.** Divide the number 10 into two such parts that the sum of their squares shall be $54\frac{1}{2}$.

Ans. $6\frac{1}{2}$ and $3\frac{1}{2}$.

A. What number is that which is less than one-tenth of its square by $7\frac{1}{2}$?

2. Divide 100 into two such parts that their product shall be 2211. Ans. 67 and 33.

B. What number is that which exceeds the half of its square by $\frac{3}{8}$?

3. There are two numbers in the proportion of 4 to 7, the sum of whose squares is 4160. What are the numbers?
Ans. $\pm 32$ and $\pm 56$.

C. Divide 100 into two such parts, that the product of the whole by one of the parts shall be equal to the square of the other part.

4. A person being asked how old he was, replied that the square of the number representing his age, in years, exceeded 20 times that number by 125. How old was he? Ans. 25.

D. A horse-dealer bought a number of young horses at a fair, all at the same price, for the sum of £1350. Of these he kept ten, and sold the remainder for £1306, 10s., making 2 guineas of profit on each of the horses sold. How many horses did he buy?

5. Divide the number 21 into two such parts, that the squares of those parts may be in the proportion of 9 to 16.
Ans. 9 and 12.

E. A tradesman computes that, during the time he has been in business, he has made £6300 of clear profit. His neighbour, however, who has been 3 years shorter time in business, has made the same sum, owing to his clearing £27 per annum more. How long is it since the first commenced business?

6. Divide 100 into two such parts, that their product may be equal to the difference of their squares.
Ans. 61·80340 –, and 38·19660 +.

F. If we add 5 to the smaller of two numbers, and subtract it from the greater, the product of the sum and difference will be 84; and if we add 3 to the smaller, and subtract it from the greater, the product of the sum and difference will be 90. What are the numbers?

7. A traveller, having proceeded 175 miles by railway, complained of the slowness of the train, and showed that, if it had only gone five miles per hour faster, he would have accomplished his journey in less time by an hour and three quarters. What was the rate at which the train went? Ans. 20 miles per hour.

G. The express railway-train from *A* to *B*, 350 miles apart, goes 10 miles per hour faster than the slow train, and reaches its destination 4 hours sooner. What are their respective rates?

8. What two numbers are those whose difference multiplied by the greater produces 104, and by the less, 88? Ans. ± 26 and ± 22.

H. Twenty-eight yards of cloth were purchased in two pieces, which differed both in quantity and in quality, and yet cost equal sums (viz. £9, 15s. a-piece), in consequence of each being accidentally rated at as many shillings per yard as there were yards in the other piece. What were the prices of the two pieces per yard?

9. A toyman purchased a lot of cricket-balls at 28s. the dozen, and, after retailing them at a uniform price, found that he had received six guineas for the whole, and that the sum he received for the last dozen was all clear profit. How many did he sell, and at what price?

Ans. 36 at 3s. 6d., or 18 at 7s. each.

K. Divide 49 into three parts, such that the difference of their differences may be 4, and the product of the least and greatest, 225.

10. The difference of two numbers is 4, and their product 117. What are the numbers? Ans. ± 13, and ± 9.

L. A wholesale shoemaker received an unexpected order for 990 pairs of shoes, to be finished by the end of the month. He found that, if these were divided equally among his men, each would have allotted to him 12 pairs more than he could get finished by the given time at his ordinary rate of working. He therefore engaged 54 more men, and got the work executed by the time specified. How many men had he at first?

11. The difference of two numbers is 2, and the difference of their cubes 152. What are the numbers?

Ans. ± 6, and ± 4.

M. A farmer, being asked how many sheep he had, replied, that he had 9 more than an exact number of scores, and that, if the square root of the number of sheep were subtracted from half the number of scores, the remainder would be 3. How many sheep had he?

12. A number of schoolboys, contributing equal sums, raised £6 for an exhibition of fire-works. Eight of their school-fellows, disapproving of the project, refused to contribute; but, had they joined, the others would have had sixpence a-piece less to pay in order to make up the sum. How many contributed? Ans. 40.

N. Find three numbers, in the proportion of $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$, such that the sum of the squares of the first and third shall be less than twice the square of the second by 584.

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# CHAPTER XXI.

## LITERAL COEFFICIENTS.

Letters are sometimes used as coefficients, being employed to represent numeral coefficients taken generally, or when not known.

The letters $m$, $n$, $p$, $q$, are most frequently employed for this purpose, or the capital letters $A$, $B$, $C$, &c., and occasionally others.

If we were required to add $mx$ to $nx$; treating these as ordinary algebraic quantities, they would be regarded as unlike, and the sum would be $mx+nx$; but, taking $m$ and $n$ as coefficients, $mx$ and $nx$ are like quantities, and their sum, by the rule for adding together like quantities, is $(m+n)x$. In like manner the difference of $mx$ and $nx$, when $m$ and $n$ are regarded as coefficients, is $(m \backsim n)x$.

**Def.** The sign $\backsim$, placed between two quantities, denotes their difference, when it is unknown which of the two is the greater.

**Example 1.** Multiply $mx^2-nx+p$ by $mx-n$.

$$\begin{array}{rrrr}
m x^2 & -nx & +p & \\
mx & -n & & \\
\hline
m^2x^3 & -mnx^2 & +mpx & \\
 & -mnx^2 & +n^2x & -np \\
\hline
\end{array}$$

$$\text{Ans. } m^2x^3-2mnx^2+(mp+n^2)x-np.$$

**Example 2.** Divide $ACx^2+(BC-AD)xy-BDy^2$ by $Ax+By$.

$$\begin{array}{r|l|l}
Ax+By & ACx^2+(BC-AD)xy-BDy^2 & Cx-Dy. \\
 & ACx^2+BCxy & \\
\cline{2-2}
 & \qquad -ADxy-BDy^2 & \\
 & \qquad -ADxy-BDy^2 & \\
\cline{2-2}
\end{array}$$

**Exercise 1.** Add together $mx^3+nx^2+px$, $px^3+qx^2-rx$, $qx^3-rx^2+x$, and $mx^3-nx^2-px$.

Ans. $(2m+p+q)x^3+(q-r)x^2-(r-1)x$.

A. What is the sum of $Ax^2+Bx+C$, $2Ax^2-2Bx+3C$, $-Bx^2+2Cx-D$, and $-3Bx^2-5Cx-8D$.

2. From $A + Bz - Cz^2 + Dz^3 - Ez^4$ subtract $a + bz - cz^2 - dz^3 + ez^4$.

Ans. $(A-a)+(B-b)z-(C-c)z^2+(D+d)z^3-(E+e)z^4$.

B. From $mw + nx - py - qz$, subtract $w - rx + sy - tz$.

3. Multiply $Ax^2 - Bx - C$ by $Bx - C$.

Ans. $ABx^3 - (B^2 + AC)x^2 + C^2$.

C. Multiply $x^2 + mx - n$ by $x^2 + px - q$.

D. Cube $A - Bx + Cx^2$.

4. Divide $ACx^3 + (AD - BC)x^2 - (AE + BD)x + BE$ by $Ax - B$. Ans. $Cx^2 + Dx - E$.

E. Divide $mx^4 - (n+mp)x^3 + (m+np+p)x^2 - (p^2+n)x + p$ by $mx^2 - nx + p$.

5. Given $mx - \frac{x}{n} = a$, to find $x$.

Ans. $x = na \div (mn - 1)$.

F. Given $\frac{z}{m} - \frac{z}{n} = c$, to determine $z$.

6. Given $mx + ny = a$ and $px + qy = b$, to find $x$ and $y$.

Ans. $x = \frac{qa - nb}{mq - np}$, $y = \frac{mb - pa}{mq - np}$.

G. Given $\frac{x}{m} - \frac{y}{n} = a$, and $\frac{x}{p} - \frac{y}{q} = b$, to find $x$ and $y$.

7. Given $x^2 + nx = a$, to find $x$.

Ans. $x = \frac{1}{2}(\pm\sqrt{4a + n^2} - n)$.

H. Given $y^2 - 2my = b^2$ to find $y$.

8. Given $mx^2 + nx = a$. Ans. $x = \frac{\pm\sqrt{(4am + n^2)} - n}{2m}$.

K. Given $\frac{y^2}{m} - \frac{y}{n} = c$.

9. Given $\frac{m(b^2 - x^2)}{bx} = mc - \frac{mx}{b}$. Ans. $x = \frac{b}{c}$.

L. $\frac{(a - mx)(b - nx)}{mn} = x^2 + \frac{ab}{mn} - \frac{ab}{mc} - \frac{b^2}{nc}$.

10. Given $\frac{y^6}{m^3} - y^3 = \frac{a^6}{m^3}$. Ans. $y = \sqrt[3]{\frac{m^3 \pm \sqrt{(4a^6 + m^6)}}{2}}$.

M. Given $(m-n)x^2 = px + \frac{mp}{m-n}$.

# CHAPTER XXII.

## LITERAL EXPONENTS.

Letters are sometimes placed as exponents instead of numbers, but as they represent numbers, they are subject to the same rules. Thus $a^m \times a^n \doteq a^{m+n}$, $a^m \div a^n \doteq a^{m-n}$, $(a^m)^n \doteq a^{mn}$, and $\sqrt[n]{a^{mn}} \doteq a^m$.

Exercise A. Express the following products, &c., in their simplest forms :—$a^{2m} \times a^{3m}$, $b^{4n} \div b^{3n}$, $c^{2m} \times c^{3n}$, $x^{m-n} \times x^n$, $y^{m+n} \div y^n$, $z^{m+n} \times z^{m-n}$, $z^{m+n} \div z^{m-n}$, $(d^m)^3$, $(e^{2m})^{3n}$, $\sqrt{(a^{6m})}$ and $\sqrt[m]{(x^{2m})}$.

1. Cube $a^m - b^n$. Ans. $a^{3m} - 3a^{2m}b^n + 3a^m b^{2n} - b^{3n}$.

B. Square $5x^{2m} - 4x^{3n}$.

2. Multiply $7y^{3p} + 5z^{4q}$ by $7y^{3p} - 5z^{4q}$.
Ans. $49y^{6p} - 25z^{8q}$.

C. Multiply $a^{3m} + a^{2m} + a^m + 1$ by $a^{4m} - a^{3m}$.

3. Divide $6c^{3p} - 13c^{2p} + 6c^p$ by $3c^p - 2$. Ans. $2c^{2p} - 3c^p$.

D. Divide $2x^{3m} - x^{2m}y^n - 2x^m y^{2n} - 8y^{3n}$ by $x^m - 2y^n$.

4. Divide $a^{5n} - 16a^n$ by $a^{2n} + 2a^n$.
Ans. $a^{3n} - 2a^{2n} + 4a^n - 8$.

E. Divide $25z^{4q} + 50z^{3q} - 100z^q + 200$ by $5z^{2q} - 10z^q + 10$.

5. Reduce $\dfrac{16a^{3n}b^n}{24a^{2n}b^{2n}}$, $\dfrac{c^{2m} + 2c^m z^m + z^{2m}}{c^{2m} - z^{2m}}$, and $\dfrac{x^{2n} + 3x^n + 2}{x^{3n} + 3x^{2n} + 4x^n + 2}$, to their lowest terms.

Ans. $\dfrac{2a^n}{3b^n}$, $\dfrac{c^m + z^m}{c^m - z^m}$, and $\dfrac{x^n + 2}{x^{2n} + 2x^n + 2}$.

F. The same with the following fractions; $\dfrac{245a^{5n}b^{3n}c^n}{175a^{2n}b^{4n}c^{3n}}$, $\dfrac{x^{2m} - y^{2n}}{x^{2m} - 2x^m y^n + y^{2n}}$, and $\dfrac{a^{2n} + 2a^n b^n + b^{2n}}{a^{3n} + b^{3n}}$.

6. Multiply $\dfrac{a^{2p} - b^{2q}}{a^{2p} + b^{2q}}$ by $\dfrac{a^{3p} + a^p b^{2q}}{a^{2p} - 2a^p b^q + b^{2q}}$. Ans. $\dfrac{a^{2p} + a^p b^q}{a^p - b^q}$.

G. Divide $\dfrac{45x^{2p}}{a^{2m} - b^{2n}}$ by $\dfrac{15y^{2q}}{a^{2m} + 2a^m b^n + b^{2n}}$.

7. Find the square root of $x^{4m} - 2x^{2m}y^{2n} + y^{4n} + 2x^{2m}z^{2p} - 2y^{2n}z^{2p} + z^{4p}$. Ans. $x^{2m} - y^{2n} + z^{2p}$.

H. Determine $\sqrt{(a^{6n} - 4a^{5n} - 2a^{4n} + 12a^{3n} + 9a^{2n})}$.

8. Given $x^{2n} + 2ax^n = b^2$, to find $x$.
Ans. $x = \sqrt[n]{\{\pm\sqrt{(a^2 + b^2)} - a\}}$.

K. Given $\frac{1}{2}z^{4m} - 3z^{2m} + 4 = 0$, to find $z$.

# CHAPTER XXIII. (21.)[*]

## ARITHMETICAL PROGRESSIONS.

### DEFINITIONS.

If a series of quantities increase or decrease by equal differences, it is called an *Arithmetical Progression*,—

as 7, 9, 11, 13, 15, 17;
or $16x$, $14x$, $12x$, $10x$, $8x$.

The difference between any two adjacent terms is called the *Common Difference*.

If $a$ represent the first term, $d$ the common difference, and $n$ the number of terms, any arithmetical progression will be represented thus:

$$a,\ a \pm d,\ a \pm 2d,\ a \pm 3d,\ \ldots\ a \pm (n-1)d.$$

**Exercise.** Fill up the blanks in the following arithmetical progressions:—

1, 2, —, —, —, —, 7;
2, 5, —, —, —, —, —;
—, —, —, —, —, 13, 15;
50, 45, —, —, —, —, —;
$x$, $x+2y$, ——, ——, ——;
21, 14, —, —, —, —, —;
$a+10b$, $a+5b$, ——, ——, ——;
——, ——, $x-y$, $x+y$, ——, ——.

### THEOREM.

*In any Arithmetical Progression, the Sum of the two extreme Terms is equal to the Sum of any two Terms equidistant from the Extremes, or to Double the middle Term when the Number of Terms is odd.*

**Dem.** If $a$ be put for the first term and $l$ for the last, the common difference being $d$, the progression will be expressed thus:—

$$a,\ a \pm d,\ a \pm 2d, \ldots\ldots\ldots\ldots l \mp 2d,\ l \mp d,\ l.$$

The same progression reversed, is

$$l,\ l \mp d,\ l \mp 2d, \ldots\ldots\ldots\ldots a \pm 2d,\ a \pm d,\ a.$$

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[*] The number enclosed in brackets indicates the Chapter in the third edition.

Adding together these two forms of the progression in the order of the terms, we have the sum of every pair $=a+l$; and, if the number of terms is odd, the middle term may be expressed in the one line by $a \pm md$, and in the other by $l \mp md$, the sum of the two being also $a+l$.

Note. Any *part* of an arithmetical progression is, of course, an arithmetical progression.

Exercise A. Fill up the blanks in the progression, 7, —, —, —, 19.

1. What is the middle term of an arithmetical progression of eleven terms, having its first term $156a$, and its last term $246a+10d$? Ans. $201a+5d$.

B. The second term of an arithmetical progression is $x+10y$; and the sixth term, $x+38y$. What is the fourth term?

## Problem I.

*Having given the common Difference of an Arithmetical Progression and the first Term, to find any other Term* (say the $n$th).

Rule. Multiply the common difference by 1 less than the number of the term required (*i. e.* by $n-1$), and add the product to the first term, or subtract it from it, according as the progression increases or decreases.

Dem. The *Rule* is derived immediately from the general form of a progression, employed in illustrating the Definition, viz.—

$$a,\ a \pm d,\ a \pm 2d,\ a \pm 3d, \ldots\ldots\ldots\ldots a \pm (n-1)d,$$

in which we observe that the 2d term is $a \pm 1d$; the 3d, $a \pm 2d$; the 4th, $a \pm 3d$; and consequently, in general, the $n$th, $a \pm (n-1)d$.

Example. What is the twentieth term of the progression $3a$, $5a$, $7a$, &c.?

$$3a+2a \times 19 \doteq 41a. \text{ Ans.}$$

Exercise 1. What is the fiftieth term of the arithmetical progression, 1, 3, 5, &c.? Ans. 99.

2. What is the tenth term of the progression $30c$, $27c$, $24c$, &c.? Ans. $3c$.

A. What is the eighth term of the progression 17, 32, 47, &c.?

B. What is the eleventh term of the progression $25a$, $20a$, $15a$, &c.?

C. What is the twelfth term of the series, $a$, $a+5$, $a+10$, &c.?

D. Compute the sixteenth term of the progression $x+4y$, $x+2y$, $x$, $x-2y$, &c.

### PROBLEM II.

*Having given any two Terms of an Arithmetical Progression, and the Number of intermediate Terms, to find the Common Difference.*

RULE. Divide the difference of the two given terms by 1 more than the number of intermediate terms: the quotient will be the common difference.

DEM. If the first of the two given terms be called $a$; the last, $l$; the com. dif., $d$; and the whole number of terms from $a$ to $l$ inclusive, $n$, the number of intermediate terms being $n-2$; then, by Pr. I., $l \doteq a \pm (n-1)d$.

$$\therefore (n-1)d \doteq l-a,$$
$$\text{and } d \doteq (l-a) \div (n-1),$$

$(n-1)$ being 1 more than the number of intermediate terms.

EXERCISE 1. The first term of an arithmetical progression is 1; and the tenth, 46. What is the common difference? Ans. 5.

A. The fifth term is 63, and the eighth 96. What is the common difference?

B. The second term is $x+10y$; and the sixth $x+38y$. Determine the common difference.

C. Complete the progression, 85, —, —, —, —, 175, by inserting the blank terms.

### PROBLEM III.

*To find the Sum of an Arithmetical Progression.*

RULE. Multiply half the sum of the first term and the last by the number of terms.

DEM. Expressing the progression, and reversing it as in the Theorem above, and putting also $s$ for the sum, and $n$ for the number of terms, we have—

First, $s = a + (a \pm d) + (a \pm 2d) \ldots\ldots + (l \mp 2d) + (l \mp d) + l$,
and, $s = l + (l \mp d) + (l \mp 2d) \ldots\ldots + (a \pm 2d) + (a \pm d) + a$.

Adding the two equations together, and observing that, when the upper sign of the two is used in the first equation, the upper is also used in the lower, and *vice versa*, we have

$$2s = (a+l) + (a+l) + (a+l), \text{ to } n \text{ terms}, = (a+l) \times n.$$

$$\therefore s = \frac{(a+l) \times n}{2} \text{ or } \frac{a+l}{2} \times n.$$

This will perhaps be more easily understood, if we take a particular case, as the progression 7, 10, 13, &c., to 14 terms. Then

$$s = 7 + 10 + 13 \ldots\ldots\ldots + 40 + 43 + 46.$$
$$\text{and } s = 46 + 43 + 40 \ldots\ldots\ldots + 13 + 10 + 7.$$
$$\therefore 2s = 53 + 53 + 53 \ldots\ldots\ldots + 53 + 53 + 53;$$
$$\text{or } 2s = 53 \times 14.$$
$$\therefore s = 53 \times 7 = 371.$$

EXERCISES. Sum the following progressions :—

1. 1, 2, 3, 4, . . . . . . . 100. Ans. 5050.

A. $17c$, $15c$, $13c$, $11c$, $9c$, $7c$.

B. $a$, $a+b$, $a+2b$, . . . . $a+10b$.

2. Find the sum of 21 terms of the series, $x$, $\frac{3x}{2}$, $2x$, $\frac{5x}{2}$, &c. Ans. $126x$.

C. Sum the series $5a$, $5a+3b$, $5a+6b$, &c., to seven terms.

3. A young man, engaging as a farm servant, asked ten pounds a year as his wages. His employer refused to give so much at first, but offered him 5 guineas for the first year, to be raised 15 shillings every year that he remained in his place. He remained 40 years on these terms. What were his last year's wages, and what did he earn during the whole time? Ans. £34, 10s.; and £795.

D. How far would a person walk before he could pick up 100 pebbles, placed in a straight line, a yard apart, and bring them one by one to the place where he stood, which was in the same straight line, and one yard from the first pebble in the row?

## Questions Relating to Arithmetical Progressions, Resolved by Equations.

**Example 1.** The first term of an arithmetical progression is 8; the last, 43; and the sum of the terms, 204. What is the common difference?

The first term, the last, the common difference, the number of terms, and their sum, being represented, as before, by the letters $a$, $l$, $d$, $n$, and $s$, we have, from Pr. III.,

$$s = \frac{a+l}{2} \times n, \text{ or } 204 = \frac{51}{2} \times n.$$

$$\therefore n = 204 \times \frac{2}{51} = 8.$$

Then, by Pr. II., $d = 35 \div 7 = 5$, Ans.

**Example 2.** The sum of five numbers in arithmetical progression is 90; and the sum of their squares, 1710. What are the numbers?

Let $x$ represent the middle term, and $y$ the common difference. Then the five numbers will be represented by $x-2y$, $x-y$, $x$, $x+y$, and $x+2y$; and their squares will be $x^2-4xy+4y^2$, $x^2-2xy+y^2$, $x^2$, $x^2+2xy+y^2$, and $x^2+4xy+4y^2$. The sum of the five numbers is $5x$; and the sum of their squares, $5x^2+10y^2$.

$$\therefore 5x = 90,\ x = 18, \text{ and } 5x^2 = 1620.$$

$$\text{But } 5x^2 + 10y^2 = 1710,$$

$$\text{or } 1620 + 10y^2 = 1710.$$

From this we obtain $y = 3$; and the five terms, 12, 15, 18, 21, and 24.

**Example 3.** There are four numbers in arithmetical progression, such that the product of the second and third is 2250; and that of the first and fourth 2200. What are the numbers?

Let the numbers be denoted by $x-3y$, $x-y$, $x+y$, and $x+3y$, the common difference being $2y$.

$$\text{Then, } x^2 - y^2 = 2250,$$

$$\text{and } x^2 - 9y^2 = 2200.$$

Hence we find $y^2 = 6\frac{1}{4}$, $2y = 5$, and $x = 47\frac{1}{2}$. Therefore the four numbers are 40, 45, 50, and 55.

Note. It will be seen, from the preceding examples, that there are various ways of expressing the terms, suitable to different questions. The best methods are soon discovered after a little experience. The object of the peculiar modes adopted in Examples 2 and 3, is, it will be perceived, to get rid of the product $xy$, in all the equations. Both questions, however, might have been resolved almost equally easily by expressing the terms, $x$, $x + y$, $x + 2y$, &c.

In the preceding examples, and in the following exercises, the progressions are understood to be ascending, unless otherwise expressed; but the processes are almost exactly the same in the case of descending progressions.

EXERCISE A. The first term is 10; and the last, 38. The sum of the terms is 120. What is the number of terms?

1. The sum of 20 terms is 200; and the last term, 29. What is the first? Ans. $-9$.

B. The first term is 77; and the last, 17. The sum of all the terms is 517. What is the second term?

C. Of three numbers in arithmetical progression, the difference of the squares of the first and second is 20; and of the second and third, 28. What is the progression?

D. The sum of the four terms of an arithmetical progression is 80, and the sum of their squares 1620. What are the terms?

E. Of five numbers in arithmetical progression, the product of the first and fifth is 48; and that of the second and fourth, 60. What is the progression?

2. The common difference is 8; the number of terms, 9; and the product of the first and last, 2112. What is the first term? Ans. 24.

F. In an arithmetical progression of four terms, the sum of the squares of the two mean terms is 1409; and that of the two extremes is 1445. Compute the numbers.

3. The product of four numbers in arithmetical progression is 98560, their common difference being 6. What is the first term? Ans. 10.

G. Of six numbers in arithmetical progression, the common difference is 4, and the ratio of the first to the last is that of 2 to 3. What are the numbers?

# CHAPTER XXIV. (22).

## GEOMETRICAL PROGRESSIONS.

### DEFINITIONS.

If a series of quantities increase, from continual multiplication by the same quantity, or decrease, from continual division by the same quantity, the series is called a *Geometrical Progression.*

as 2, 6, 18, 54, 162, 486;
or 64, 32, 16, 8, 4, 2.

Even when the series decreases it may be considered to do so by multiplication, the multiplier, in that case, being a fraction. Consequently every geometrical progression may be expressed thus,—

$$a, ar, ar^2, ar^3, \ldots\ldots\ldots\ldots\ldots\; ar^{n-1}.$$

Regarding all geometrical progressions in the latter point of view, viz. as formed by successive multiplications, the constant multiplier, $r$, is called the *common Ratio* of the terms. As usual $n$ expresses the number of terms.

Exercise A. What are the values of $r$ and $n$ in each of the two numerical progressions given above?

B. Form a geometrical progression of five terms, having its first term $7a$, and the common ratio of the terms 2.

C. Form a geometrical progression of four terms, having its first term $4x$, and the common ratio $5y$.

D. The first term of a geometrical progression is $y$, the second $y \div 3$, and the last $x$. What is the last but one?

E. If the second and third terms are $\frac{x}{y}$ and $\frac{y}{x}$, what are the first and fourth?

Note. The successive terms of a geometrical progression are continual proportionals: that is,—the first is to the second as the second to the third, as the third to the fourth, and so on: for $a : ar :: ar : ar^2 :: ar^2 : ar^3$, and so on. Consequently any term, standing between two others, is a mean proportional between them.

Exercise F. Fill up the blanks in the following geometrical progressions:—

3, —, 12, —, —, —.
—, —, $9a$, —, $81a$.

### THEOREM.

*In any Geometrical Progression the Product of the two extreme Terms is equal to the Product of any two Terms equidistant from them; and also equal to the Square of the middle Term when the Number of Terms is odd.*

DEM. If the first term be $a$ and the last $l$, the progression may be expressed thus:—

$$a,\ ar,\ ar^2,\ \ldots\ldots\ldots\ldots,\ \frac{l}{r^2},\ \frac{l}{r},\ l.$$

Then $a \times l \doteq al$, $ar \times \frac{l}{r} \doteq al$, $ar^2 \times \frac{l}{r^2} \doteq al$, and so on. If there is an odd number of terms, the middle term may be expressed as either $ar^m$ or $\frac{l}{r^m}$; and again, $ar^m \times \frac{l}{r^m} \doteq al$.

Or, if the progression be expressed thus,—

$$a,\ ar,\ ar^2,\ \ldots\ldots\ldots\ldots\ldots\ ar^{n-3},\ ar^{n-2},\ ar^{n-1}, \text{ then}$$
$$a \times ar^{n-1} \doteq a^2r^{n-1},\ ar \times ar^{n-2} \doteq a^2r^{n-1},\ ar^2 \times ar^{n-3} \doteq a^2r^{n-1},$$

and so on, all the products being alike. If there is an odd number of terms, the middle term will be $ar^{\frac{n-1}{2}}$, the square of which is $a^2r^{n-1}$, as before.

NOTE. Any part of a geometrical progression is, of course, a geometrical progression.

EXERCISE 1. In an increasing geometrical progression, consisting of seven terms, the first is 13, and the middle one 104. What is the last? Ans. 832.

2. If the first term is $7a$, and the fifth $567a^5b^4$, what are the intermediate terms? Ans. $21a^2b$, $63a^3b^2$, and $189a^4b^3$.

A. In a geometrical progression the first term is $\frac{c}{b}$, the fifth $\frac{2}{3}$, and the sixth $1\frac{1}{2}$. What is the tenth?

B. The first term of a geometrical progression is $3x$, and the fifth term $\frac{4x^5}{3}$. What is the ninth?

### PROBLEM I.

*Having given the first Term of a Geometrical Progression, and the common Ratio, to find any other Term* (say the *n*th).

RULE. Raise the common ratio to a power whose exponent is a unit less than the number of the term required, viz. $n-1$, and multiply that power into the first term.

**Dem.** The rule follows immediately from the definition of a geometrical progression. For, by that definition, if the first term be $a$, the common ratio $r$, and the number of terms $n$, the progression will be

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots\ldots\ldots\ldots,\ ar^{n-1},$$

in which any term is equal to the product of $a$ into that power of $r$ whose index is 1 less than the number of the term.

**Example.** The first term of a geometrical progression being 50, and the common ratio 7, what is the fourth term?

$$7^3 \times 50 \doteq 343 \times 50 = 17150. \text{ Ans.}$$

**Exercise 1.** The first term of a geometrical progression is 13; and the common ratio, 2. What is the tenth term?
Ans. 6656.

2. The common ratio being $\frac{c}{3}$, and the first term $\frac{27}{c^3}$, what is the seventh term? Ans. $\frac{c^3}{27}$.

A. The first term of a geometrical progression being ·0016, and the common ratio $5a$, what is the tenth term?

B. The first term being $\frac{e^4}{16}$, and the common ratio $\frac{2}{e}$, what is the ninth term?

3. What is the amount of £488, for four years, at 5 per cent. per annum, compound interest? Ans. £593·16705.

C. What is the amount of £200, for 5 years, at 4 per cent. per annum, compound interest?

## Problem II.

*Having given the first Term of a Geometrical Progression, the last Term, and the common Ratio, to find the Sum of all the Terms.*

**Rule.** Divide the difference of the first term and the last by the difference of the common ratio and unity. To the quotient add the last term.

**Dem.** Expressing the progression in the general form shown in the Definitions, and putting $s$ for the sum of the terms,

$$s \doteq a + ar + ar^2 \ldots\ldots\ldots\ldots + ar^{n-2} + ar^{n-1}, \ldots\ldots\ldots (1),$$

$$\therefore rs \doteq \quad ar + ar^2 \ldots\ldots\ldots\ldots + ar^{n-2} + ar^{n-1} + ar^n,\ (2),$$

every term in Eq. 1 being multiplied by $r$, but placed in Eq. 2 so as to be adjacent to the corresponding term in Eq. 1. That being done, let us subtract Eq. 1 from Eq. 2, and we have

$$rs - s, \text{ or } (r-1)s = ar^n - a.$$

$$\therefore s = \frac{ar^n - a}{r-1} = ar^{n-1} + \frac{ar^{n-1} - a}{r-1},$$

the last expression being obtained from the previous one by dividing the numerator by the denominator.

Now $ar^{n-1} - a$ is the difference of the first term and the last, and $r-1$ is the difference of the common ratio and unity. The quotient of these two differences, viz. $\frac{ar^{n-1}-a}{r-1}$ added to the last term, $ar^{n-1}$, is the expression for $s$, the sum of the terms.

Example. The first term of a geometrical progression is $8a$, and the common ratio 3, what is the sum of seven terms.

By Pr. I. the seventh term is $8a \times 3^6 = 8a \times 729 = 5832a$.

$$\frac{5832a - 8a}{3-1} = \frac{5824a}{2} = 2912a.$$

$$2912a + 5832a = 8744a, \text{ Ans.}$$

Exercise 1. The first term of a geometrical progression is 3, and the common ratio 2 : what is the sum of ten terms? Ans. 3069.

2. What is the sum of a geometrical progression of eight terms, whose first term is 413,343, and common ratio, $\frac{1}{3}$? Ans. 619,920.

A. What is the sum of the geometrical progression $x$, $xy$, $xy^2$, ........., $xy^6$, supposing $x = 256$, and $y = 1\frac{1}{2}$?

B. A man laid by a farthing for his son when he was a year old, a halfpenny the year after, a penny the third year, and so on,—intending to continue doubling the sum every year till he came of age. If he did so, what would be the accumulated amount which the son would receive at the end of his 21st year?

Note. By the same rule we may find the sum of a *decreasing* geometrical progression, even *when the number of terms is infinite*, if we regard the last term as $= 0$.

Exercise 3. What is the sum of the series, $1 + \frac{1}{2} + \frac{1}{4} +$ &c. *ad infinitum*? Ans. 2.

4. What is the sum of the progression, $1 + \frac{2}{3} + \frac{4}{9} +$ &c. *ad infinitum*? Ans. 3.

C. Sum the series, $1+\frac{1}{3}+\frac{1}{9}+$ &c. *ad infinitum.*

D. The same with the series, $a+\frac{3}{4}a+\frac{9}{16}a+$ &c. without end.

## QUESTIONS RELATING TO GEOMETRICAL PROGRESSIONS, RESOLVED BY EQUATIONS.

EXAMPLE 1. Of four numbers in geometrical progression, the difference of the first and last is 21; and that of the second and third, 6. What are the numbers?

Let $x$ and $y$ represent the second and third terms. Then since, as has been shown before, all the terms of a geometrical progression are in continued proportion, we have

$$\text{First term} : x :: x : y.$$

$$\text{Therefore the first term} = x^2 \div y.$$

In like manner, since $x : y :: y :$ last term, we have, last term $= y^2 \div x$. Our progression, therefore, stands thus,—

$$\frac{x^2}{y},\ x,\ y,\ \frac{y^2}{x}.$$

If then, we regard it as a decreasing progression, we have, by the conditions of the question,

$$\frac{x^2}{y}-\frac{y^2}{x}=21,$$

$$\text{and } x-y=6.$$

$$\text{From the first eq., } x^3-y^3=21xy.$$

$$\text{From the second, } \quad x-6=y.$$

Dividing the third equation by the second,

$$x^2+xy+y^2=\tfrac{7}{2}xy,$$

$$\text{or } x^2-\tfrac{5}{2}xy+y^2=0.$$

Inserting $x-6$ for $y$ in this equation, we obtain

$$x^2-6x=72.$$

Resolving this as a quadratic equation, we find

$$x=12 \text{ or } -6.$$

$$\therefore y=x-6=6 \text{ or } -12.$$

Therefore the progression is either 24, 12, 6, 3, or $-3$, $-6$, $-12$, $-24$; that is, regarding it as a decreasing progression. But if we take it as an increasing one, and proceed in the same manner, we find the progression either 3, 6, 12, 24, or $-24$, $-12$, $-6$, $-3$.

NOTE. We have thus shown, in one instance, the various results on the hypothesis of the progression being either ascending or descending, and consisting either of positive or negative terms. But in the questions which follow, we shall no longer take the same latitude, but assume that the progressions consist only of positive numbers, and that they are always increasing progressions, unless otherwise described.

EXAMPLE 2. The sum of three numbers in geometrical progression is 189; and the sum of their squares, 22113. Determine the numbers.

Let $x$ be the first term, and $y$ the common ratio. Then the progression will be $x$, $xy$, $xy^2$.

$$\therefore x + xy + xy^2 = 189;$$

$$\text{and } x^2 + x^2y^2 + x^2y^4 = 22113.$$

From Eq. 1, $\quad x + xy^2 = 189 - xy.$

Squaring this, and then subtracting $x^2y^2$ from both sides,

$$x^2 + x^2y^2 + x^2y^4 = 35721 - 378xy.$$

From this subtract Eq. 2, and we have

$$0 = 13608 - 378xy,$$

$$\therefore xy = 36, \text{ and } x = 36 \div y.$$

Insert 36 for $xy$ in Eq. 1, and simplify the result.

$$x + 36y = 153.$$

$$\text{That is, } \frac{36}{y} + 36y = 153.$$

$$\text{Hence we find } 4y^2 - 17y = -4.$$

Resolving this quadratic, we find $y = 4$ or $\frac{1}{4}$. But limiting our question to the case of an ascending progression, we take 4 as the value of $y$. Hence $x = 36 \div y = 9$, and the progression is 9, 36, 144.

EXERCISE A. In an increasing geometrical progression of three terms, the sum of the first and second terms is 28; and that of the second and third, 84. What are the terms?

B. Determine four numbers in geometrical progression, such that the difference of the first and second shall be 24; and that of the third and fourth, 384.

1. Of three numbers in geometrical progression, the sum is 114, and the sum of the first and second, is to that of the third and second, as 4 to 28. Determine the numbers.

Ans. 2, 14, and 98.

2. The difference of the first and second terms is $\frac{1}{9}$; and that of the second and third is $\frac{1}{3}$. What are the three terms?

Ans. $\frac{1}{18}$, $\frac{1}{6}$, and $\frac{1}{2}$.

C. Of five numbers in geometrical progression, the product of the first and fifth is 2025; and the sum of the second and fourth, 150. Find the numbers.

*D.* Four numbers are required, in geometrical progression, such that the first shall be to the fourth as 8 to 27, and the sum of the four shall be $2\frac{11}{27}$.

3. Of three numbers in geometrical progression the sum is 10·4; and the sum of the first and second is to that of the first and third, as 2 to 5. Ascertain the numbers.
Ans. ·8, 2·4, and 7·2.

4. The sum of the first and third terms of a geometrical progression exceeds the second term by 9; and the sum of the squares of the three terms is 189. What is the progression? Ans. 3, 6, 12.

5. Find the four terms of a geometrical progression, such that the sum of the first and fourth shall be 126, and that of the second and third, 30. Ans. 1, 5, 25, 125.

# CHAPTER XXV.

## ALGEBRA APPLIED TO GEOMETRY.

Algebra is sometimes applied to Geometry for the investigation of theorems expressed in general terms, or for the discovery and expression of general rules for the computation of the lengths of lines, the areas of surfaces, or the contents of solids. At other times it is used for the solution of problems by means of letters representing unknown quantities. It is only to the latter of these purposes that it will be applied in this chapter.

Before the scholar enters upon this department of Algebra, he ought to go over the second Part of this Course, —on Practical Geometry and Mensuration. No other rules are required here.

Example 1. The hypothenuse of a right-angled triangle is 117, and the sum of the sides containing the right angle 153. What are the sides?

Put $x$ and $y$ to represent the sides: then, from Part II. Ch. II. Pr. 67, $x^2+y^2 \doteq 117^2$; and, from the remaining datum of the question, $x+y \doteq 153$.

Doubling the first of these equations, and subtracting from that the square of the second, we have

$$x^2-2xy+y^2=3969.$$
$$\therefore x-y \doteq \pm 63.$$

Adding this to the second equation, and taking it from it, we find $2x = 216$ or 90, and $2y = 90$ or 216. $\therefore$ the one side is 108 and the other 45.

EXERCISE 1. The base of a right-angled triangle is 55, and the difference of the hypothenuse and perpendicular is 25. What are the hypothenuse and perpendicular?

Ans. 73 and 48.

A. The perpendicular of a right-angled triangle is 39, and the sum of the hypothenuse and base is 169. What is the base?

2. The hypothenuse of a right-angled triangle is 65, and the difference of the other two sides 13. What are the sides? Ans. 39 and 52.

B. The perimeter of a rectangle is 92, and the diagonal 34. What are its length and breadth?

3. The area of a rectangle is 1848 and its diagonal 65. What is its length?

Ans. 56.

C. The area of a rectangle is 91 and its perimeter 40. What is its length?

4. A straight line drawn from one corner of a square to the middle of the opposite side is 5 yards in length. What is the side of the square?

Ans. $2\sqrt{5}$.

D. The altitude of an isosceles triangle is 12 and the sum of the three sides 36. What are the sides?

5. A footpath passes round two sides of a square field from one corner of the field to the opposite corner. It is found, by measurement, that it would save fifty yards of walking if the footpath cut diagonally across the field. What is the length of the side?

Ans. $50 + 25\sqrt{2}$, or $85{\cdot}355+$ yards.

E. One square is inscribed in another in such a manner that each corner of the inner square is twice as far from the one extremity of the side on which it rests as from the other, and the difference of the areas of the two squares is 100. What is the area of the inner square?

6. One square is inscribed in another in such a position that the areas of the two are in the ratio of 16 to 25, the side of the larger square being 10. What is the distance of each corner of the inner square from each extremity of that side of the outer square on which it rests?

Ans. $5 \pm \sqrt{7}$.

F. The area of a right-angled triangle is 30, and the sum of its three sides is also 30. What is the hypothenuse?

7. A rectangular garden is surrounded by a hedge and ditch, the *whole* rectangular space thus occupied being 55 yards in length and covering quarter of an acre, 360 square yards of which are taken up by the hedge and ditch. What is the united width of these two?

Ans. $2\frac{1}{2}$ yards.

G. The circumference of a hexagon in lineal feet and its area in square feet are expressed by the same number. What is the length of one side?

8. A ladder, being placed against a wall, reached the sill of a window 52 feet from the ground. Upon its foot being brought 14 feet nearer the wall, it reached another window sill 8 feet higher than the former. What was the length of the ladder? Ans. 65 feet.

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# CHAPTER XXVI. (23).

## PROMISCUOUS EXERCISES.

A. Add together $\frac{2}{3}a-\frac{1}{2}b+\frac{5}{6}c$, $\frac{1}{2}a-\frac{3}{4}b-\frac{2}{3}c$, and $\frac{9}{10}a-\frac{3}{8}b-\frac{1}{4}c$.

1. Find the greatest common measure of $18x^3-53xy^2-28y^3$, and $36x^4-48x^3y-56xy^3-49y^4$.

Ans. $6x^2-8xy-7y^2$.

2. Add together $\dfrac{x^2-14x+49}{x^2+14x+49}$, $\dfrac{x^2+49}{x^2-49}$, and $\dfrac{x^2+14x+49}{x^2-14x+49}$.

Ans. $\dfrac{3x^4+588x^2+2401}{x^4-98x^2+2401}$.

B. Multiply $\dfrac{a^2+b^2}{a-b}$ by $\dfrac{a^2-b^2}{a^3+a^2b+ab^2+b^3}$.

C. Divide $4x-12$ by $\dfrac{1}{2}\sqrt{(x-3)}$.

D. Raise $x-\sqrt{x}$ to the fourth power.

3. A man and his wife found that a cheese, of the ordinary size used by them, usually lasted a fortnight when they were both at home; but when the husband was absent, it lasted his wife 5 weeks and a day. How long would it serve the man when his wife was absent?

Ans. $22\frac{10}{11}$ days.

E. Extract the square root of $4y^4-12y^3\sqrt{y}+29y^3-30y^2\sqrt{y}+25y^2$.

F. Given $\dfrac{4x-25}{2\sqrt{x}+5}=\sqrt{x}-2$, to find $x$.

G. There are three villages, *A*, *B*, and *C*, situated triangularly with regard to each other. A letter-carrier goes round them once a-day, and finds that, when he walks at his usual pace of 3 miles an hour, the last stage from *C* to *A* takes him 20 minutes less time than the first, from *A* to *B*, but 50 minutes more than the intermediate stage, from *B* to *C*, and that, after deducting the time lost in stoppages, it takes him just 6 hours to go all round. What are the distances of the villages from one another?

H. What is the sum of $+\dfrac{1}{2a^2+2ax}$, $+\dfrac{1}{2a^2-2ax}$, and $-\dfrac{1}{a^2+x^2}$?

4. What two numbers are those whose sum exceeds their difference by 7, and whose difference exceeds the smaller of the two by two-thirds? Ans. $7\frac{2}{3}$ and $3\frac{1}{2}$.

5. Given $\dfrac{10z-18}{\sqrt{(5z)}+3}-2=\sqrt{(5z)}-3$. Ans. $z=5$.

K. A farmer sold a number of sheep and cattle, in the proportion of seven of the former to two of the latter. The cattle were sold, on an average, at 8 guineas a-piece, and the sheep at 25 shillings. After paying 15s. 6d. of expenses, he brought away £305, 16s. 6d. How many had he of each kind?

6. Three pumps are occasionally employed for filling a certain reservoir with water. The first and second, when worked together, can fill the cistern in 70 minutes; the first and third can fill it in 105 minutes; and the second and third, in 140 minutes. In what time can the reservoir be filled by each of the three pumps, when worked alone?
Ans. In 120, 168, and 840 minutes, respectively.

7. Divide, $\dfrac{x^2+3x-10}{x^2-x-6}$ by $\dfrac{x^2+x-20}{x^2+x-12}$.

Ans. $\dfrac{x^2+2x-8}{x^2-2x-8}$.

8. One eighth of the inhabitants of a town were cut off by a pestilence, during the first year of its visitation, and by death from other causes; but 851 were added by births and influx from other places. On the second year, the *tenth part* of the inhabitants perished while 1031 were

again added to their number. After this the population of the town was found to be exactly the same as it had been before the pestilence broke out. What was the number? Ans. 8456.

L. Divide 238 into two parts which shall be to each other in the proportion of $\frac{2}{3}$ to $\frac{3}{4}$.

M. The scale of the Centigrade thermometer has its zero at the freezing point of water, corresponding to 32° of Fahrenheit, and is divided into 100 degrees between the freezing and the boiling points, so that the 100th degree of the Centigrade corresponds to the 212th of Fahrenheit. At what point on the scale is the same temperature indicated by the same number on each?

N. A gentleman bought separately two contiguous fields, containing together 40 acres. Each of these cost as many pounds per acre as there were acres in the field; and the prices of the two fields were in the proportion of 4 to 9. What were the areas of the two fields?

9. A number, consisting of two digits, is reversed, and the reverse subtracted from the original number. It is then found, that the remainder is less than half the given number by 6, but equal to three times the sum of its digits. Ans. 84.

O. Given $x^3 - y^3 = 98$, and $x - y = 2$, to find $x$ and $y$.

10. Two men and a boy concerted to steal a parcel of money, and, having succeeded, met to divide the contents. One of the men, who divided it, took to himself twice as much as he gave to the boy; and, to keep the other man quiet, gave him a half crown more than would have been his share if the money had been equally divided among the three, but, at the same time 5 shillings less than he took to himself? What was the whole sum stolen? Ans. £4, 2s. 6d.

11. Given $x - \sqrt{x} = 1$. Ans. $x = \frac{1}{2}(3 \pm \sqrt{5})$.

P. The fifth term of a geometrical progression is $16875a^{11}$; and the common ratio, $5a^2$. What is the first term?

Q. What sum, laid out at 4 per cent. compound interest, for six years, will amount to £1000?

12. Determine the five terms of an ascending geometrical progression, such that the product of the second and fourth shall be 13689, and that the difference of the first and fifth shall be 1040.

Ans. $+13, +39, +117, +351, +1053$,
or $-1053, -351, -117, -39, -13$.

R. If the hour and minute hands of a clock are together at 12, what ought to be their next time of meeting?

13. Two women occupy adjacent stalls at a fair. The one makes four sales for the other's three, and has sold three dozen articles before the other commences; but then she only draws as much at three sales as her neighbour does at two. How many articles must the latter dispose of before she have as much money in hand as the former? Ans. 216.

S. The periodic times of Mercury and Venus are respectively 88 and 225 days. How many days intervene between two conjunctions of these two planets?

14. A certain descending geometrical progression consists of four terms, the continued product of which is 186624, the difference of the second and third terms being 24. What are the terms? Ans. 108, 36, 12, and 4.

15. Two railway trains were dispatched from the same station, and in the same direction. The second started when the electric telegraph announced that the first was just leaving a station 30 miles in advance, and set out at such a speed that it went 5 miles to the other's 3. After continuing all the 30 miles at that rate, it diminished its speed by 14 miles an hour, making its rate now to that of the train before it as 6 to 5. Suppose both should continue afterwards at the same velocities, and have no stoppages, how long would the second train be, after its first starting, before it overtook the first train? Ans. 3 h. 36 m.

T. A traveller, before the time of railways, set out from Edinburgh to London, a distance of 389 miles,—advancing, for the first hundred miles, at the rate of $7\frac{1}{2}$ miles per hour, and for the remainder at 8 miles. At the same time, another traveller set out from London to Edinburgh,—proceeding at the rate of 10 miles per hour for the first hundred miles, and the rest of the way at 9 miles. Find at what distance from London they would meet.

16. Having given $d$, the difference between the diagonal and side of a square, to find the side.

Ans. Side $= d(1+\sqrt{2})$.

17. A ladder, when placed with its foot 25 feet from a wall, reaches to the top; but it does not reach within 8 feet of that height when the foot is removed 14 feet further back. What is the length of the ladder?

Ans. 65 feet.

U. In a right-angled triangle, given the base, $b$, and the sum of the hypothenuse and perpendicular, $s$, to find these separately.

V. Having given the diagonal, $d$, of a rectangle, and the sum, $s$, of its length and breadth, to find the sides.

W. Having given $s$, the side of a hexagon, to find the radius of a circle inscribed within it.

# CHAPTER XXVII.

## DETACHED COEFFICIENTS.

When quantities consisting of several terms are arranged according to the powers of one or more letters, it saves trouble, in subjecting them to the processes of multiplication, division, involution and evolution, if, throughout the operation, we put down the coefficients only, with their signs, and, when the work is finished, attach the letters with their exponents. It will be seen at a glance what ought to be the exponents of the letters in the first term of the result, and also the order in which the different powers must follow in the subsequent terms.

EXAMPLE 1. Multiply $x^3-3x^2y+3xy^2-y^3$ by $x^2-2xy+y^2$.

$$\begin{array}{rrrrrr}
1 & -3 & +3 & -1 & & \\
1 & -2 & +1 & & & \\
\hline
1 & -3 & +3 & -1 & & \\
 & -2 & +6 & -6 & +2 & \\
 & & +1 & -3 & +3 & -1 \\
\hline
1 & -5 & +10 & -10 & +5 & -1.
\end{array}$$

We have thus got the coefficients of the product, and, since the first term must evidently be $x^5$ and the other terms must follow according to the descending powers of $x$ and the ascending powers of $y$, the exponents differing by a unit in each term, the result of the whole operation will be $x^5-5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5$.

EXERCISE 1. Multiply $-4a^2+10ab-25b^2$ by $-4a^2-10ab-25b^2$. Ans. $16a^4+100a^2b^2+625b^4$.

A. What is the square of $5x^4-8x^3+6x^2-8x+5$?

2. Multiply $y-3y^2+9y^3$ by $1+3y$. Ans. $y+27y^4$.

B. Multiply $x^2+6xy+9y^2$ by $x^2-6xy+9y^2$.

EXAMPLE 2. What is the continued product of $x-1$, $x-2$, $x-3$, and $x-4$?

$$\begin{array}{rrrrr}
1 & -1 & & & \\
1 & -2 & & & \\
\hline
1 & -1 & & & \\
 & -2 & +2 & & \\
\hline
1 & -3 & +2 & & \\
1 & -7 & +12 & & \\
\hline
1 & -3 & +2 & & \\
 & -7 & +21 & -14 & \\
 & & +12 & -36 & +24 \\
\hline
1 & -10 & +35 & -50 & +24. \\
\hline\hline
\end{array}
\qquad
\begin{array}{rrr}
1 & -3 & \\
1 & -4 & \\
\hline
1 & -3 & \\
 & -4 & +12 \\
\hline
1 & -7 & +12 \\
\end{array}$$

Ans. $x^4-10x^3+35x^2-50x+24$.

EXERCISE 3. Multiply together $x-3$, $x+3$, $x^2+3x+9$, and $x^2-3x+9$. Ans. $x^6-729$.

C. What is the sixth power of $a-b$?

D. What is the continued product of $y-2$, $y-3$, $y-4$, and $y-5$?

NOTE 1. If, in any of the given quantities, the terms are arranged according to the powers of the letters, but with one or more powers wanting, we place a cipher as the exponent of that power. Thus, if we had given $x^3+5x-5$, we should write the coefficients $1+0+5-5$.

EXERCISE 4. Square $a^3-3a+6$.

Ans. $a^6-6a^4+12a^3+9a^2-36a+36$.

E. Multiply $x^4-3xy^3+8y^4$ by $x^3+3x^2y-y^3$.

NOTE 2. But if, in *all* the given quantities, the powers omitted are in regular order from first to last, so that the exponents of the powers, in all the given terms, are in arithmetical progression, with the same difference in each of the expressions, then it is not necessary to supply ciphers as the coefficients of the omitted terms, since the exponents in the result will also be in an arithmetical progression with the same given difference.

EXERCISE 5. What is the cube of $x^4+2x^2-1$?

Ans. $x^{12}+6x^{10}+9x^8-4x^6-9x^4+6x^2-1$.

F. What is the square of $a^6 - 2a^3b^3 + 3b^6$?

G. Multiply $27a^6 + 2a^4b^2 + 9a^2b^4 + b^6$ by $27a^6 - 27a^4b^2 + 9a^2b^4 - b^6$.

**EXAMPLE 3.** Divide $x^4 - 6x^2 + 16x - 15$ by $x^2 - 2x + 3$.

$$\begin{array}{rl}
1-2+3\big) & 1+0-6+16-15\big(1+2-5 \\
 & \underline{1-2+3} \\
 & \quad +2-9+16 \\
 & \quad \underline{+2-4+\ \ 6} \\
 & \qquad -5+10-15 \\
 & \qquad \underline{\underline{-5+10-15}}. \quad \text{Ans. } x^2+2x-5.
\end{array}$$

**NOTE 3.** In dividing, it may always be known when the operation is finished by marking that term in the dividend which contains the same power of the leading letter as occurs in the first term of the divisor. When that term has disappeared from the remainder, the quotient is completed, and the terms then remaining must be annexed to the quotient in a fractional form. Other easy methods will be found in particular cases.

**EXERCISE 6.** Divide $6x^4 - 19x^3y + 25x^2y^2 - 17xy^3 - 4y^4$ by $2x^2 - 3x + 4$.

Ans. $3x^2 - 5x - 1$.

7. Divide $16a^4 + 100a^2b^2 + 625b^4$ by $+4a^2 - 10ab + 25b^2$.

Ans. $4a^2 + 10ab + 25b^2$.

8. Divide $32a^6 + 46a^4 - 68a^2 - 98$ by $8a^3 + 10a^2 + 12a + 14$.

Ans. $4a^3 - 5a^2 + 6a - 7$.

H. Divide $25x^4 + 50x^2 - 100x + 200$ by $5x^2 - 10x + 10$.

K. Divide $729x^6 - 243x^4y^2 + 27x^2y^4 - y^6$ by $27x^3 + 27x^2y + 9xy^2 + y^3$.

L. Divide $a^6 - 117649$ by $a^2 - 7a + 49$.

9. Divide $32a^5 - 240a^4b + 720a^3b^2 - 1100a^2b^3 + 800ab^4 - 240b^5$ by $8a^3 - 36a^2b + 54ab^2 - 27b^3$.

Ans. $4a^2 - 12ab + 9b^2 - \dfrac{20a^2b^3 + 10ab^4 - 3b^5}{8a^3 - 36a^2b + 54ab^2 - 27b^3}$.

M. Divide $x^5 - 4x^3 - 20x$ by $x^3 - 2x^2 + 3x - 4$.

10. Find the greatest common measure of $a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5$ and $a^3 - a^2x - ax^2 + x^3$.

Ans. $a^2 - 2ax + x^2$.

N. Reduce $\frac{c^4+6c^3y-3c^2y^2-10cy^3-2y^4}{c^4+4c^3y-13c^2y^2+6cy^3+2y^4}$ to its lowest terms.

**Example** 4. What is the square root of $a^4-4a^3b+6a^2b^2-4ab^3+b^4$?

$$\begin{array}{l)l}
1 & 1-4+6-4+1\,(1-2+1 \\
1 & 1 \\ \hline
2-2 & -4+6 \\
\;\;-2 & -4+4 \\ \hline
2-4\;+1 & 2-4+1 \\
 & \underline{\underline{2-4+1}} \quad \text{Ans. } a^2-2ab+b^2
\end{array}$$

**Exercise** 11. Extract the square root of $a^6-4a^5-2a^4+12a^3+9a^2$. Ans. $a^3-2a^2-3a$.

12. Of $9x^6-12x^5+10x^4-28x^3+17x^2-8x+16$? Ans. $3x^3-2x^2+x-4$.

O. Of $25e^8-40e^7+46e^6-44e^5+35e^4-20e^3+10e^2-4e+1$.

P. Of $49a^4-70a^3b+67a^2b^2-30ab^3+9b^4$.

# CHAPTER XXVIII.

## EVOLUTION.

(*Continued from Chapter* VIII.)

### Problem I.

*Having given any compound Quantity not an exact Square, to find its Square Root in the Form of an infinite Series.*

**Rule.** Continue the process described in Chapter VIII., Problem II., until the law of the series appears, or until a sufficient approximation to the true root has been obtained.

**Note.** By *the law of the series* is meant the way in which *each term* is deduced from the previous term.

**EXAMPLE 1.** What is the square root of $a^2+b^2$?

$$
\begin{array}{r|l}
a & a^2+b^2\left(a+\dfrac{b^2}{2a}-\dfrac{b^4}{8a^3}+\dfrac{b^6}{16a^5}-\dfrac{5b^8}{128a^7}+\&c.\right. \\
a & a^2 \\
\hline
2a+\dfrac{b^2}{2a} & +b^2 \\
\dfrac{b^2}{2a} & +b^2+\dfrac{b^4}{4a^2} \\
\hline
2a+\dfrac{b^2}{a}-\dfrac{b^4}{8a^3} & -\dfrac{b^4}{4a^2} \\
-\dfrac{b^4}{8a^3} & -\dfrac{b^4}{4a^2}-\dfrac{b^6}{8a^4}+\dfrac{b^8}{64a^6} \\
\hline
2a+\dfrac{b^2}{a}-\dfrac{b^4}{4a^3}+\&c. & +\dfrac{b^6}{8a^4}-\dfrac{b^8}{64a^6} \\
 & +\dfrac{b^6}{8a^4}+\dfrac{4b^8}{64a^6}-\&c. \\
\cline{2-2}
 & -\dfrac{5b^8}{64a^6}+\&c.
\end{array}
$$

The root thus found will assume a perfectly regular form if we separate the coefficients into factors, introducing one or two common factors into the numerators and corresponding denominators, thus—

$$a+\frac{b^2}{2a}-\frac{1.b^4}{2.4a^3}+\frac{1.3b^6}{2.4.6a^5}-\frac{1.3.5.b^8}{2.4.6.8a^7}+\&c.$$

**EXERCISE 1.** Extract the square root of $1+x$.

$$\text{Ans. } 1+\frac{1}{2}x-\frac{1}{2.4}x^2+\frac{1.3}{2.4.6}x^3-\frac{1.3.5}{2.4.6.8}x^4+\&c.$$

A. Compute the square root of $1-y^2$.

2. Expand $\surd(x^2+y)$ into an infinite series.

$$\text{Ans. } x+\frac{y}{2x}-\frac{1.y^2}{2.4x^3}+\frac{1.3y^3}{2.4.6x^5}-\frac{1.3.5y^4}{2.4.6.8x^7}+\&c.$$

B. The same with $\surd(x^2-2y^2)$.

**NOTE.** The process is similar to the above, for extracting the square root of a *number* when the root does not come out exactly. In that case it is usually carried out as an interminate decimal; but it may also, if we choose, be found in the form of a series of fractions. Thus

$$\surd 2=\surd(1+1)\doteq 1+\frac{1}{2}-\frac{1}{2.4}+\frac{1.3}{2.4.6}-\frac{1.3.5}{2.4.6.8}+\&c.$$

### THEOREM.

*The* mn$^{\text{th}}$ *root of any Quantity may be found by extracting the* m$^{\text{th}}$ *Root of the* n$^{\text{th}}$ *Root of that Quantity.*

Thus $\sqrt[6]{a} \doteq \sqrt[2]{\sqrt[3]{a}}$.

DEM. Since $b \doteq \sqrt{b^2}$ and $b^2 = \sqrt[3]{b^6}$, it follows that $b \doteq \sqrt{\sqrt[3]{b^6}}$. But $b \doteq \sqrt[6]{b^6}$. $\therefore \sqrt[6]{b^6} \doteq \sqrt{\sqrt[3]{b^6}}$. For $b^6$ write $a$. Then $\sqrt[6]{a} \doteq \sqrt{\sqrt[3]{a}}$, or $\sqrt[2]{\sqrt[3]{a}}$.

In the same manner it may be proved generally that $\sqrt[mn]{a} \doteq \sqrt[m]{\sqrt[n]{a}}$.

### PROBLEM II.

*To extract the fourth Root of any compound Quantity.*

RULE. Extract the square root of the given quantity and then the square root of that.

The *demonstration* is given in the preceding theorem.

EXERCISE 1. What is the fourth root of $a^4 - 12a^3 + 54a^2 - 108a + 81$? Ans. $a - 3$.

A. What is the fourth root of $x^8 - 8x^6y^3 + 24x^4y^6 - 32x^2y^9 + 16y^{12}$?

2. Compute the fourth root of $\frac{1}{16}a^4 - \frac{1}{2}a^2b^2 + b^4$.

Ans. $\frac{1}{2}a - \frac{b^2}{a} - \frac{b^4}{a^3} - \frac{2b^6}{a^5} - \frac{5b^8}{a^7} -$ &c.

B. Expand $\sqrt[4]{\left\{\frac{x^4}{y^4} + 2 + \frac{y^4}{x^4}\right\}}$ into a series.

COR. In like manner the *eighth* root may be obtained by three successive extractions of the square root; the *sixteenth* root by four such operations, and so on.

### PROBLEM III.

*To extract the Cube Root of any given compound Quantity.*

CASE 1. *When the given quantity is an exact cube, and that the cube of a binomial.*

RULE. In this case we may represent the required root by $x \pm y$, $x$ and $y$ denoting either single letters, or single terms made up of several factors. The cube of $x \pm y$ being $x^3 \pm 3x^2y + 3xy^2 \pm y^3$, the *given* quantity must be in this form if it is an exact cube, and $x$, the first term of the root, will be the cube root of $x^3$, the first term of the given power. The second term of the root will be found by dividing the second term of the given power, viz. $\pm 3x^2y$, by 3 times the square of the first term of the root, the terms of the given quantity having, of course, been arranged according to the powers of the letters.

To ascertain if the cube of the root thus found has exhausted the whole of the given quantity (which it *must* do *if* that quantity is a complete cube), we must make up successively the terms $\pm 3x^2y + 3xy^2 \pm y^3$, by inserting, in these, for $x$ and $y$, the values found for them. If these terms agree with the corresponding terms of the given quantity, that quantity is proved to be an exact cube, and we have found its root correctly.

The rule contains its own *demonstration.*

EXAMPLE 1. Extract the cube root of $8a^3 - 36a^2b + 54ab^2 - 27b^3$.

$$\begin{array}{r l}
 & 8a^3 - 36a^2b + 54ab^2 - 27b^3\big(2a - 3b, \text{ Ans.} \\
 & 8a^3 \\
\hline
12a^2\big) & -36a^2b + 54ab^2 - 27b^3 \\
 & -36a^2b + 54ab^2 - 27b^3 \\
\hline\hline
\end{array}$$

EXERCISE 1. What is the cube root of $125c^3 - 300c^2d + 240cd^2 - 64d^3$? Ans. $5c - 4d$.

A. Of $8x^3 - 6x^2y + \frac{3}{2}xy^2 + \frac{1}{8}y^3$.

CASE 2. *When the given quantity is an exact cube, with its root consisting of more than two terms.*

RULE. Since the root may be expressed by $(x \pm y) \pm z$ ($z$ representing one or more terms), the cube of which is $(x \pm y)^3 \pm 3(x + y)^2z + 3(x + y)z^2 + z^3$, it follows that every cube of a compound quantity contains a quantity which may be expressed by $(x \pm y)^3$, and, if the terms are suitably arranged, commencing with that quantity. If that quantity is in the *form* $(x \pm y)^3$, we have at once $x \pm y$ as two terms of the root. But if expanded, as it usually is, into the form $x^3 \pm 3x^2y + 3xy^2 \pm y^3$, we obtain the two first terms by Case 1.

Having found these we regard $(x \pm y)$ as *one* term, and proceed to find another term as directed in Case 1; and so on until all the terms have been found.

**Example 2.** What is the cube root of $a^6-6a^5+21a^4-44a^3+63a^2-54a+27$?

$$\begin{array}{rl}
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad (a^2-2a+3. \\
 & a^6-6a^5+21a^4-44a^3+63a^2-54a+27 \\
 & a^6 \\ \cline{2-2}
3a^4\big) & -6a^5+21a^4-44a^3 \\
 & -6a^5+12a^4-\phantom{0}8a^3 \\ \cline{2-2}
3a^4-12a^3+12a^2\big) & +\ 9a^4-36a^3+63a^2-54a+27 \\
 & +\ 9a^4-36a^3+36a^2 \\
 & \qquad\qquad\quad +27a^2-54a+27 \\ \hline\hline
\end{array}$$

**Exercise 2.** Find the third root of $27x^3-54x^2y+27x^2z+36xy^2-36xyz+9xz^2-8y^3+12y^2z-6yz^2+z^3$.

Ans. $3x-2y+z$.

B. The same of $a^6-a^5+\frac{5}{27}a^3-\frac{1}{81}a-\frac{1}{729}$.

**Case 3.** *When the given quantity is not an exact cube.*

**Rule.** Commence by the same process as that described for Case 1 or Case 2, and, when all the terms of the given quantity have been brought down, proceed in the same manner with the remainder, continuing the operation as far as may be required, or carrying out the root into the form of an infinite series.

**Exercise 3.** What is the cube root of $x^3-3x^2y$?

Ans. $x-y-\frac{y^2}{x}-\frac{5y^3}{3x^2}-$ &c.

C. What is the cube root of $1+x$, carried out to five terms?

**Cor.** The *sixth* root may be found by extracting the square root, and then the cube root of that; the *twelfth* root by extracting the cube root of the fourth root, and so on with other powers whose exponents are powers of 3 or products of powers of 2 and 3.

# CHAPTER XXIX.

## QUADRATIC EQUATIONS.

(*Continued from Chapter* XIX.)

**Note 1.** A quadratic equation, of the form $m^2x^2 \pm nx = a$, may be resolved without removing the coefficient of the first term, by regarding $mx$ as the unknown quantity instead of $x$, and proceeding by the general rule for adfected quadratics. That is,—divide $n$ by $2m$ and add the square of the quotient to both sides of the given equation : extract the square root and transpose the second term : then divide by $m$.

The reason of this is obvious, for, after the addition directed, the left-hand member of the equation will evidently be a complete square, since it is of the form $m^2x^2 \pm nx + \left(\frac{n}{2m}\right)^2$, the first and third terms being squares and the second term being double the product of their two roots.

This method, however, has no real advantage over the usual process, viz. first dividing by the coefficient of $x^2$.

EXAMPLE 1. Resolve the equation $m^2x^2 \pm nx = a$.

$$m^2x^2 \pm nx + \left(\frac{n}{2m}\right)^2 \doteq a + \frac{n^2}{4m^2} = \frac{4m^2a + n^2}{4m^2}.$$

$$\therefore mx \pm \frac{n}{2m} \doteq \pm \frac{1}{2m} \sqrt{(4m^2a + n^2)}.$$

$$\text{Hence } x \doteq \frac{1}{2m^2} \times \{ \pm \sqrt{(4m^2a + n^2)} \mp n \}.$$

The sign of the last term, $n$, will not be + or − at pleasure, but the reverse of that which it was in the given equation.

EXERCISES. Resolve the following equations :—

1. $4x^2 + 11x = 15$. Ans. $x = +1$ or $-3\frac{3}{4}$.
A. $9y^2 - 10y = 51$.
2. $16z^2 - 7z = 365$. Ans. $z = +5$ or $-4\frac{9}{16}$.
B. $25x^2 + 16x = 464$.
3. $\frac{4}{9}y^2 + \frac{2}{3}y = 72$. Ans. $y = +12$ or $-13\frac{1}{2}$.
C. $\frac{25}{16}z^2 - \frac{15}{2}z = -5$.

NOTE 2. A quadratic equation, of the form $mx^2 \pm nx = a$, may also be resolved without dividing by $m$, if we multiply the equation by $m$ and proceed as directed in Note 1. But it is easier to multiply by $4m$ and then complete the square : we thus entirely avoid fractional coefficients until the final step.

The latter is called the *Hindoo* method. By avoiding fractions it has an appearance of simplicity, but is not in reality easier than the common method, viz. first dividing by $m$.

**Example** 2. Resolve the equation $mx^2 \pm nx = a$.

$$4m^2x^2 \pm 4mnx \doteq 4ma.$$
$$4m^2x^2 \pm 4mnx + n^2 \doteq 4ma + n^2.$$
$$\therefore 2mx \pm n \doteq \pm \sqrt{(4ma+n^2)},$$
$$\text{and } x \doteq \frac{1}{2m} \times \{ \pm \sqrt{(4ma + n^2)} \mp n \}.$$

**Exercise** 4. Given $3x^2 + 5x = 22$, to determine the value of $x$. Ans. $x = 2$ or $-3\frac{2}{3}$.

D. Resolve the equation $2y^2 - 7y = 15$.

5. Resolve $5z^2 - 8z = 624$. Ans. $z = +12$ or $-10{\cdot}4$.

E. Resolve $7x^2 + 4x = 276$.

**Note** 3. A quadratic equation, of the form $\frac{1}{x^2} \pm \frac{m}{x} = a$, may be resolved without multiplying by $x^2$, by regarding $\frac{1}{x}$ as the unknown quantity, finding its value by the general method, and taking the reciprocal of that as the value of $x$.

**Example** 3. Resolve the general equation expressed in Note 3.

$$\frac{1}{x^2} \pm m.\frac{1}{x} \doteq a.$$
$$\frac{1}{x^2} \pm m.\frac{1}{x} + \frac{m^2}{4} \doteq \frac{4a+m^2}{4}.$$
$$\therefore \frac{1}{x} \pm \frac{m}{2} \doteq \pm \frac{\sqrt{(4a+m^2)}}{2},$$
$$\text{or } \frac{1}{x} = \frac{\pm \sqrt{(4a+m^2)} \mp m}{2},$$
$$\text{and } x = \pm \frac{2}{\sqrt{(4a+m^2)} - m}.$$

**Exercise** 6. Given $\frac{1}{x^2} + \frac{4}{x} = 2\frac{1}{4}$, to find the value of $x$.

Ans. $x = +2$ or $-\frac{2}{9}$.

F. Given $\frac{1}{y^2} - \frac{1}{y} + \frac{4}{25} = 0$, to find $y$.

7. Given $\frac{1}{z^2} - \frac{3}{z} + 2 = 0$. Ans. $2 = +\frac{1}{2}$ or $+1$.

G. Given $\frac{3}{x^2} + \frac{5}{x} = 42$.

**Note** 4. In resolving quadratics involving two unknown quantities, such an expression as $x^2y^2$ or $\frac{x^2}{y^2}$ may be treat*ed as the* square of a single unknown quantity, as of $xy$ or

$\frac{x}{y}$; and, when the value of that is found from the one equation, the separate values of $x$ and $y$ may be found from that value and the other equation.

EXAMPLE 4. Given $x^2 + y^2 = 61$, and $x^2y^2 - 2xy = 840$, to find the values of $x$ and $y$.

Commencing with the second equation and treating $xy$ as the unknown quantity, we find $xy = +30$ or $-28$.

Adding the double of this to the first equation and subtracting it from it, we find

$$x^2 + 2xy + y^2 = 121 \text{ or } 5,$$
$$\text{and } x^2 - 2xy + y^2 = 1 \text{ or } 117,$$
$$\therefore x + y = \pm 11 \text{ or } \pm\sqrt{5},$$
$$\text{and } x - y = \pm 1 \text{ or } \pm\sqrt{117}.$$
$$\text{Hence } x = \pm 6 \text{ or } \pm 5 \text{ or } \pm\tfrac{1}{2}\sqrt{5} \pm \tfrac{1}{2}\sqrt{117},$$
$$\text{and } y = \pm 5 \text{ or } \pm 6 \text{ or } \pm\tfrac{1}{2}\sqrt{5} \mp \tfrac{1}{2}\sqrt{117}.$$

It will be observed, however, that, although we obtain so many values of $x$ and $y$, we are not at liberty to connect them together in any way we please. In the first place, when $xy = 30$, we obtain only the rational values, and when $xy = -28$ we obtain only the surd values. A surd value of $x$ cannot therefore co-exist with a rational value of $y$, or the converse. Again, when $x + y = +11$, $x$ can only be $+6$ and $+5$, and $y$ can only be $+5$ and $+6$; and when $x + y = -11$, then both values of $x$ and both values of $y$ must be negative: a positive rational value of the one letter cannot therefore be taken at the same time with a negative rational value of the other. For a similar reason we cannot have $x$ and $y$ of the *same* value: thus $x$ cannot be $= +6$ and $y = +6$ at the same time.

In the same way, in regard to the surd values, it may be shown that according as the value of $x$ has a positive or a negative sign attached to $\sqrt{5}$, so that of $y$ must also have. Along with either of these, however, the sign of $\sqrt{117}$ may either be positive or negative in the value of $x$, provided that it be the contrary in the value of $y$.

Our answer must therefore, to be strictly correct, be expressed in a more guarded form thus:—

Ans. $x = +6$ and $y = +5$, or $x = +5$ and $y = +6$,
or $x = -6$ and $y = -5$, or $x = -5$ and $y = -6$;
or $x = +\frac{1}{2}\sqrt{5} + \frac{1}{2}\sqrt{117}$ and $y = +\frac{1}{2}\sqrt{5} - \frac{1}{2}\sqrt{117}$,
or $x = +\frac{1}{2}\sqrt{5} - \frac{1}{2}\sqrt{117}$ and $y = +\frac{1}{2}\sqrt{5} + \frac{1}{2}\sqrt{117}$,
or $x = -\frac{1}{2}\sqrt{5} + \frac{1}{2}\sqrt{117}$ and $y = -\frac{1}{2}\sqrt{5} - \frac{1}{2}\sqrt{117}$,
or $x = -\frac{1}{2}\sqrt{5} - \frac{1}{2}\sqrt{117}$ and $y = -\frac{1}{2}\sqrt{5} + \frac{1}{2}\sqrt{117}$.

EXERCISE 8. Given $x + y = 12$, and $x^2y^2 + 4xy = 480$.

Ans. $x \doteq 10$ and $y = 2$, or $x = 2$ and $y = 10$;
or $x \doteq 6 + 2\sqrt{15}$ and $y = 6 - 2\sqrt{15}$,
or $x = 6 - 2\sqrt{15}$ and $y = 6 + 2\sqrt{15}$.

H. Given $y - z = 1$, and $y^2z^2 + 16yz = 2436$.

9. Given $x + z = 20$, and $\frac{x^2}{z^2} - 6\frac{x}{z} = -9$.

Ans. $x = 15$, $z = 5$.

K. Given $3x^2 - 5y^2 = 28$, and $\frac{4x^2}{y^2} - \frac{8x}{y} = -3$.

L. Given $x^2y^2 + 5xy = 1476$, and $\frac{x^2}{y^2} - \frac{5x}{y} = -4$.

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# CHAPTER XXX.

## QUADRATIC AND CUBIC EQUATIONS.

### PROBLEM I.

*Having given an Equation in the Form* $\mathbf{x^2 + ax = b}$, *or* $\mathbf{x^3 + ax^2 + bx = e}$, *to transform it into another Equation in which the second Term shall be wanting.*

RULE. For $x$ substitute, in the quadratic equation, $y - \frac{1}{2}a$, in the cubic, $y - \frac{1}{3}a$, and we shall have equations in the forms $y^2 = B$, $y^3 + By = C$.

DEM. When $x \doteq y - \frac{1}{2}a$,

$$\begin{array}{rl} x^2 \doteq & y^2 - ay + \frac{1}{4}a^2, \\ \text{and } ax \doteq & \phantom{y^2} + ay - \frac{1}{2}a^2 \\ \hline \therefore x^2 + ax \doteq & y^2 \ldots\ldots - \frac{1}{4}a^2 = b, \\ \text{and } & y^2 = \frac{1}{4}a^2 + b. \end{array}$$

When $x \doteq y - \frac{1}{3}a$,

$$\begin{array}{rl} x^3 \doteq & y^3 - ay^2 + \frac{1}{3}a^2y - \frac{1}{27}a^3, \\ ax^2 = & \phantom{y^3} + ay^2 - \frac{2}{3}a^2y + \frac{3}{27}a^3, \\ bx = & \phantom{y^3 + ay^2 - \frac{2}{3}a^2y} by - \frac{1}{3}ab. \\ \hline \end{array}$$

$$\therefore x^3 + ax^2 + bx \doteq y^3 + (b - \tfrac{1}{3}a^2)\,y + \tfrac{2}{27}a^3 - \tfrac{1}{3}ab = e.$$

$$\therefore y^3 + (b - \tfrac{1}{3}a^2)\,y \doteq -\tfrac{2}{27}a^3 + \tfrac{1}{3}ab + e.$$

COR. The same principle is of general application to equations of every degree. That is, if we substitute $y - \frac{a}{n}$ *for* $x$ in the expression $x^n + ax^{n-1} + bx^{n-2} +$ &c., the second term disappears; but the proof will be better understood after the binomial theorem has been demonstrated.

NOTE. If, in the given equation, $a$ is negative, or, to be more clear, if the equation is in the form $x^2 - ax = b$, or $x^3 - ax^2 + bx = c$, then for $x$ we must substitute $y + \frac{1}{2}a$, or $y + \frac{1}{3}a$, for $-\frac{1}{2}(-a)$ and $-\frac{1}{3}(-a)$ are really $+\frac{1}{2}a$ and $+\frac{1}{3}a$.

EXAMPLE 1. Transform the equation, $x^3 - 6x^2 + 4x = 100$, into another equation of the same degree in which there shall be no term containing the second power of the unknown quantity.

For $x$ substitute $y + 2$. Then

$$\begin{array}{rcrrrr}
x^3 & = & y^3 & +6y^2 & +12y & +\ 8, \\
-6x^2 & = & & -6y^2 & -24y & -24, \\
+4x & = & & & +\ 4y & +\ 8. \\
\hline
\therefore x^3 - 6x^2 + 4x & = & y^3 & & -8y & -\ 8 \;= 100,
\end{array}$$

$$\text{or } y^3 - 8y = 108.$$

EXERCISE 1. Transform the equation $x^2 + 12x = 86$ into a pure quadratic. Ans. $y^2 = 50$.

A. The same with $x^2 - 10x = -21$.

2. The same with $x^2 - 5x = 66$. Ans. $y^2 = 72\frac{1}{4}$.

B. The same with $x^2 + 9x = 162$.

3. Transform the equation $x^3 - 3x^2 + 6x = 8$ into another cubic equation without the second power of the unknown quantity. Ans. $y^3 + 3y = 4$.

C. The same with $x^3 - 12x^2 + 20x = -105$.

4. The same with $x^3 + 5x^2 - 11x - 21 = 0$.
Ans. $y^3 - 19\frac{1}{3}y + 6\frac{16}{27} = 0$.

D. The same with $x^3 + 10x^2 - 10x = 184$.

COR. By this means any mixed quadratic may be converted into a pure quadratic and resolved without completing the square.

EXAMPLE 2. Find the value of $x$ in the equation, $x^2 - 16x + 63 = 0$.

Substituting $y + 8$ for $x$, we find $y^2 = 1$. $\therefore y = \pm 1$, and $x = y + 8 = 9$ or $7$.

EXERCISE 5. Resolve the equation $x^2 + 8x = 65$, without completing the square. Ans. $x = 5$ or $-13$.

E. The same with $x^2 - 22x + 105 = 0$.

6. The same with $x^2 - 11x = 42$. Ans. $x = 14$ or $-3$.

F. The same with $x^2 + x = 0{\cdot}75$.

7. The same with $x^4 - 14x^2 = 32$.
Ans. $x = \pm 4$ or $\pm \sqrt{(-2)}$.

G. The same with $x^6 + 74x^3 = 2727$.

## PROBLEM II.

*To resolve a pure Cubic Equation involving only one unknown Quantity.*

DEF. A cubic equation is said to be *pure* when it contains no other power of the unknown quantity besides the third. The same term, however, is applied when the one side of the equation is an exact cube whose root is not the unknown quantity itself but some multiple or part of it, or the sum or difference of it (or of a multiple or part of it) and a known quantity.

RULE. When the equation has been reduced to its proper form,—that is, to the form $x^3 = c$, $(ax)^3 = c$, $(x \pm b)^3 = c$, or $(ax \pm b)^3 = c$, we have then merely to extract the cube root of each side, and we obtain the value of $x$, $ax$, $x \pm b$, or $ax \pm b$.

EXERCISE 1. Resolve the equations $x^3 = 1728$, and $27y^3 = 125$. Ans. $x = 12$, and $y = 1\frac{2}{3}$.

A. Resolve $x^3 = 729$, and $64y^3 = 343$.

2. Given $z^3 - 9z^2 + 27z - 27 = 0$, to find the value of $z$. Ans. $z = 3$.

B. Given $x^3 + 6x^2 + 12x + 8 = 343$.

3. Given $8y^3 + 36y^2 + 54y + 27 = 216$. Ans. $y = 1\frac{1}{2}$.

C. Given $125z^3 - 300z^2 + 240z - 64 = 4096$.

NOTE. If the given equation is of the form $ax^3 = c$, or $a(x \pm b)^3 = c$, we have only to divide by $a$, and we obtain a pure cubic equation in one of the forms given in the rule.

## PROBLEM III.

*To resolve a mixed Cubic Equation involving only one unknown Quantity.*

CASE 1. *When the left hand member of the equation is not an exact cube, but exceeds or comes short of that by some known quantity.*

RULE. If that quantity be taken from, or added to, each side of the equation, the left hand member will become an exact cube, and its root may be found by Chapter XXVIII. Problem III.

**Example 1.** Given $x^3 - 3x^2 + 3x - 65 = 0$, to find $x$.

Here the left hand member would have been a complete cube if the last term had been $-1$ instead of $-65$. If, therefore, we add 64 to both sides, we have $x^3 - 3x^2 + 3x - 1 = 64$. Hence, extracting the cube root, $x - 1 \doteq 4$, and $x \doteq 5$.

**Exercise 1.** Resolve the equation $x^3 + 15x^2 + 75x + 25 \doteq 1231$. Ans. $x = 6$.

A. Given $y(4y^2 - 6y + 3) = 63$, to find $y$.

**Case 2.** *When the given equation is of the form $y^3 + cy = d$, the second term being absent.*

**Rule.** For $y$ substitute $z - \frac{c}{3z}$, and we obtain an equation containing only the sixth and third powers of $z$. Treating that equation as a quadratic (Ch. XIX. Note 2), we find the value of $z^3$, and thence that of $z$. Having found the value of $z$ we substitute it for $z$ in the equation $y = z - \frac{c}{3z}$, and we obtain the value of $y$.

**Dem.** Since $y \doteq z - \frac{c}{3z}$, we have

$$y^3 = z^3 - cz + \frac{c^2}{3z} - \frac{c^3}{27z^3}, \text{ and}$$

$$cy = \quad + cz - \frac{c^2}{3z}. \quad \therefore$$

$$y^3 + cy \doteq z^3 - \frac{c^3}{27z^3} = d, \text{ and, consequently,}$$

$$z^6 - \frac{c^3}{27} = dz^3, \text{ or } z^6 - dz^3 = \frac{c^3}{27}.$$

Resolving this as a quadratic, we find

$$z^3 = \tfrac{1}{2}d \pm \tfrac{1}{2}\sqrt{(\tfrac{4}{27}c^3 + d^2)}, \text{ and}$$

$$z = \sqrt[3]{\{\tfrac{1}{2}d \pm \tfrac{1}{2}\sqrt{(\tfrac{4}{27}c^3 + d^2)}}.$$

**Example 2.** Given $y^3 + 36y = 37$.

For $y$ substitute $z - \frac{12}{z}$. Then

$$y^3 \doteq z^3 - 36z + \frac{432}{z} - \frac{1728}{z^3}.$$

$$36y \doteq \quad + 36z - \frac{432}{z}.$$

$$\therefore y^3 + 36y \doteq z^3 - \frac{1728}{z^3} = 37.$$

$$\text{Hence } z^6-37z^3 \doteq 1728, \text{ and}$$
$$z^6-37z^3+(18\tfrac{1}{2})^2=2070{\cdot}25.$$
$$\therefore z^3 \doteq 64 \text{ or } -27, \text{ and } z=4 \text{ or } -3.$$
$$\therefore y=z-\frac{12}{z} \doteq 4-3, \text{ or } -3+4, =1.$$

**Note** 1. In resolving the preceding example, it may have been observed that, although the solution of the quadratic equation gives us two values of $z^3$ and consequently two values of $z$, yet $y$, or $z-\frac{12}{z}$, has only one value. It is always so in resolving equations of the form $y^3+cy=d$; for, if we put $z'$ for the first value of $z$, and $z''$ for the second, we find $z'$ is always $=-\frac{c}{3z''}$, and, consequently, $z''=-\frac{c}{3z'}$. Therefore $z-\frac{c}{3z} \doteq z''-\frac{c}{3z''}$, each of these expressions being $=z'+z''$.

But, if it be asked how it happens that $z' \doteq -\frac{c}{3z''}$ in all instances, it may be observed, from the general expression for $z^3$ contained in the demonstration, that $(z')^3 \doteq \frac{1}{2}d+\frac{1}{2}\sqrt{(\frac{4}{27}c^3+d^2)}$, and $(z'')^3 \doteq \frac{1}{2}d-\frac{1}{2}\sqrt{(\frac{4}{27}c^3+d^2)}$; and, consequently, $(z')^3 \times (z'')^3 \doteq \frac{1}{4}d^2-\frac{1}{4}(\frac{4}{27}c^3+d^2)=\frac{1}{27}c^3$. $\therefore$ $(z')^3 \doteq \left(\frac{c}{3z''}\right)^3$ and $z'=\frac{c}{3z''}$.

A similar remark is applicable to Case 3.

**Exercise** 2. Given $y^3+30y=117$, to find the value of $y$. Ans. $y=3$.

B. Resolve the equation $x^3+42x=335$.

**Def.** The sign $>$ means *greater than*, and $<$ signifies *less than*. Thus 7 is $>$ 6, and 6 is $<$ 7.

**Case** 3. *When the given equation is of the form* $y^3-cy=d$.

**Rule.** For $y$ substitute $z+\frac{c}{3z}$, and proceed as in Case 2.

**Note** 2. The rule for this case is exactly the same as that for the preceding, since $y^3-cy$ is identical with $y^3+3(-c)z$, and $-\frac{-c}{3z}$ is the same as $+\frac{c}{3z}$.

**Note** 3. In this case the general expression for the value of $z$ is $\sqrt[3]{\{\frac{1}{2}d \pm \frac{1}{2}\sqrt{(-\frac{4}{27}c^3+d^2)}}$. If, then, $d^2$ is $>\frac{4}{27}c^3$,

$-\frac{4}{27}c^3 + d^2$ is a positive quantity and has a real square root. But, if $d^2$ is $< \frac{4}{27}c^3$, $-\frac{4}{27}c^3 + d^2$ is a negative quantity and has no square root, and therefore the value of $z$ cannot be found although $z + \frac{c}{3z}$, or $y$, is a real quantity. When this occurs, the value of $y$ cannot be obtained by this rule, or by any other method except by trials or by approximation.

EXERCISE 3. Resolve the equation $y^3 - 12y = 65$.
Ans. $y = 5$.

C. Resolve the equation $y^3 - 48y = 520$.
D. Given $y^3 - 24y = 72$, to find $y$.
4. Given $x^3 + 9x = 4\frac{5}{8}$. Ans. $x = \frac{1}{2}$.
E. Resolve $x^3 - 15x = 1000{\cdot}125$.

EXAMPLE 3. Given $y^3 + 5y = 40$, to determine $y$.

Substituting $z - \frac{5}{3z}$ for $y$, we find $z^3 - \frac{125}{27z^3} = 40$, and $\therefore$ $z^6 - 40z^3 = \frac{125}{27}$.

$$\therefore z^6 - 40z^3 + 400 \doteq \frac{10925}{27} = \frac{32775}{81}.$$

$$\therefore z^3 - 20 \doteq \pm \tfrac{1}{9}(181{\cdot}03867) = \pm 20{\cdot}1154 + .$$

$$\text{Hence } z^3 \doteq +40{\cdot}1154 + \text{ or } -(0{\cdot}1154 +);$$

$$\text{and } z \doteq +3{\cdot}4234 + \text{ or } -(0{\cdot}4869 \pm).$$

Taking the first of these values for $z$, the second will, by Note 1, be $= -\frac{5}{3z}$.

$$\therefore y = z - \frac{5}{3z} \doteq 2{\cdot}936 + .$$

EXERCISE 5. Given $y^3 + 3y = 100$. Ans. $y = 4{\cdot}43 -$.
F. Given $x^3 - 2x = 12$, to find the value of $x$.

NOTE 4. The value of $z$ may contain an irrational quantity, and yet that of $y$ be rational. The reason is, that the general expression for the value of $z^3$, viz. $\frac{1}{2}d \pm \frac{1}{2}\sqrt{(\pm \frac{4}{27}c^3 + d^2)}$ is sometimes an exact cube, having its root in the form $g \pm \sqrt{h} = z$. We have therefore $z' = g + \sqrt{h}$, and $z'' = \pm\frac{c}{3z'} = g - \sqrt{h}$ (see Note 1). $\therefore y = z' + z'' \doteq 2g$, the surd disappearing.

This will be better understood from the following example. The general principle on which it depends will be more fully developed, and the conditions, under which such a result is possible, defined, in a subsequent chapter.

Example 4. Given $y^3 - 12y = 144$.

For $y$ substituting $z + \frac{4}{z}$, we have $z^3 + \frac{64}{z^3} = 144$; from which we find

$$\begin{aligned} z^3 &= 72 \pm \sqrt{5120} = 72 \pm (71{\cdot}5542\ -) \\ &= 143{\cdot}554 + \text{ or } 0{\cdot}4458 +. \\ \therefore z &\doteq 5{\cdot}236 + \text{ or } 0{\cdot}764 -. \\ \therefore y &= z + \frac{4}{z} \doteq (5{\cdot}236 +) + (0{\cdot}764 -) = 6. \end{aligned}$$

If we adhere to surd forms instead of proceeding by decimals, we resume the equation

$$\begin{aligned} z^3 &= 72 \pm \sqrt{5120} = 72 \pm 32\sqrt{5}. \\ &= 27 \pm 27\sqrt{5} + 45 \pm 5\sqrt{5}. \\ &= 3^3 \pm 3.3^2\sqrt{5} + 3.3\,(\sqrt{5})^2 \pm (\sqrt{5})^3. \end{aligned}$$

This expression is an exact cube, the root of which is $3 \pm \sqrt{5}$. $\therefore z' \doteq 3 + \sqrt{5}$, and $z'' = \frac{4}{z'} \doteq 3 - \sqrt{5}$. See Note 1.

$$\therefore y = z' + \frac{4}{z'} \doteq (3 + \sqrt{5}) + (3 - \sqrt{5}) = 6.$$

Exercise 6. Given $y^3 - 6y = 40$. Ans. $y = 4$.
G. Resolve the equation $x^3 + 12x = 112$.

Note 5. If the given equation is of the form $ay^3 \pm ey = f$, we have merely to divide by $a$, and, if of the form $y^3 \pm cx \pm e = f$, we must transpose $e$. We then have the equation of the form $y^3 \pm cy = d$. Sometimes both operations may be required.

Exercise 7. Given $216x^3 + 1080x = 215999$.
Ans. $x = 9\frac{5}{6}$.

H. $3y^3 - 126y - 153 = 900$.

Case 4. *When the given equation is of the form* $\mathbf{x^3 \pm ax^2 \pm bx = e}$.

Rule. We must change it into the form $y^3 \pm cy = d$, by Problem I., and then determine the value of $y$ by Case 2 or 3. From that we obtain the value of $x = y \mp \frac{1}{3}a$.

Note 6. Sometimes, in removing the second term from the given equation the third also disappears, $c$ becoming $= 0$, and the transformed equation taking the form $y^3 = d$. *In this* case we find the value of $y$ by Case 1.

**Example** 5. Given $x^3 + 3x^2 - 9x = 27$.
For $x$ substitute $y - 1$, and we have $y^3 - 12y = 16$.
For $y$ substitute $z + \frac{4}{z}$, and we find $z^3 + \frac{64}{z^3} = 16$.
From this we obtain $z^3 = 8$ and $z = 2$.
Hence $y = 2 + \frac{4}{2} \doteq 4$, and $x = y - 1 \doteq 3$.

**Exercise** 8. Given $x^3 - 12x^2 + 48x = 280$. Ans. $x = 10$.
K. Resolve $x^3 + 18x^2 + 108x = 513$.
9. Given $x^3 - 3x^2 + 6x = 8$. Ans. $x = 2$.
L. Given $x^3 + 6x^2 + 3x = 38$, to find $x$.
10. Given $x^3 + 6x^2 + 16x = -8\frac{8}{27}$. Ans. $x = -\frac{2}{3}$.
M. Resolve the equation $x^3 - 4x^2 + 12x = 1296$.

**Note** 7. If the first term have a coefficient, the equation must first be divided by that coefficient, and, if a known quantity stand as a separate term on the left hand side, it must be transposed, to bring the equation under Case 4.

**Exercise** 11. Given $\frac{1}{3}x^3 + 2\frac{2}{3}x^2 + 6x = 45$. Ans. $x = 3$.
N. Given $3x^3 - 6(x-1)^2 = 57$.

## CHAPTER XXXI.

## FRACTIONAL AND NEGATIVE EXPONENTS.

We have as yet made no use of negative exponents and done little more than allude to the use of fractional exponents. It is our purpose now to enter formally into both subjects, taking along with them the exponent *zero*, or 0.

**Def.** 1. When a fractional exponent, or index, is employed, the denominator indicates a root, and the numerator a power. Thus $a^{\frac{1}{5}}$ means the fifth root of $a$, and $x^{\frac{5}{7}}$ stands for the seventh root of the fifth power of $x$; but, in reading these expressions, they are named as if they were fractional *powers*, as $a$ to the one-fifth power, $x$ to the five-sevenths power.

**Cor.** 1. Since $\sqrt[3]{(x^2)} \doteq \sqrt[3]{(xx)} = \sqrt[3]{x} \times \sqrt[3]{x}$ (Ch. XVI. Pr. II.) $= (\sqrt[3]{x})^2$, $x^{\frac{2}{3}}$ may be read either as the cube root of the square of $x$ or as the square of the cube root of $x$, since the two expressions are identical; and similarly with all other fractional powers. Therefore also $\sqrt[3]{(x^2)}$ and $(\sqrt[3]{x})^2$ may either of them be written simply $\sqrt[3]{x^2}$, with no vinculum, without any risk of ambiguity.

COR. 2. Since, from the definition, quantities with fractional exponents are really surds, all the rules given for the latter, in Chapter XVI., are applicable to the former.

EXERCISE A. What is the square of $(a^5-b^5)^{\frac{1}{2}}$, the cube of $(x^2-y^2)^{\frac{1}{3}}$, and the squares of $5x^{\frac{1}{2}}$ and $2(7z)^{\frac{1}{2}}$?

B. Raise $6c^{\frac{1}{3}}$ and $2(3d)^{\frac{1}{3}}$ each to the third power.

C. Multiply $14(2x-3y)^{\frac{1}{3}}$ by $13(2x-3y)^{\frac{1}{4}}$.

D. What is the product of $2a^{\frac{1}{2}}$, $3a^{\frac{1}{3}}$, and $4a^{\frac{1}{4}}$?

E. Square $1+x^{\frac{1}{2}}$ and $x-(ay)^{\frac{1}{2}}$, separately.

F. Multiply $5a^{\frac{1}{2}}+6b^{\frac{1}{2}}$ by $5a^{\frac{1}{2}}-6b^{\frac{1}{2}}$.

G. Multiply $7a+5(bc)^{\frac{1}{2}}$ by $3a-2(bc)^{\frac{1}{2}}$.

H. Express in their simplest forms $(a^6)^{\frac{1}{2}}$, $(b^8c)^{\frac{1}{2}}$, $(64x^3y)^{\frac{1}{3}}$, $(16a^4x)^{\frac{1}{4}}$, and $(144ab^2c^4)^{\frac{1}{2}}$.

K. The same with $(300a^2x)^{\frac{1}{2}}$, $(\frac{1}{4}y)^{\frac{1}{2}}$, $\left(\frac{y}{9}\right)^{\frac{1}{2}}$, $\left(\frac{1}{125}x^3y^2\right)^{\frac{1}{3}}$, and $\left(\frac{27a}{8b^3}\right)^{\frac{1}{3}}$.

L. Multiply $x^{\frac{1}{3}}$ by $y^{\frac{1}{3}}$, $(21a)^{\frac{1}{2}}$ by $(2b)^{\frac{1}{2}}$, and $(x+3)^{\frac{1}{2}}$ by $(x-3)^{\frac{1}{2}}$.

M. What is the product of $+(2a)^{\frac{1}{3}}$, $-(3b)^{\frac{1}{3}}$, and $+(x^2)^{\frac{1}{3}}$?

1. Multiply $(20ab)^{\frac{1}{2}}$ by $(15ac)^{\frac{1}{2}}$. Ans. $10a(3bc)^{\frac{1}{2}}$.

N. Multiply $(15xy)^{\frac{1}{2}}$ by $(21yz)^{\frac{1}{2}}$, and $(4x^2)^{\frac{1}{3}}$ by $(2x)^{\frac{1}{3}}$.

O. Simplify the expression $(2abc)^{\frac{1}{2}}\times(3acd)^{\frac{1}{2}}\times(6bd)^{\frac{1}{2}}$.

P. Multiply $5(ab)^{\frac{1}{2}}$ by $6(ac)^{\frac{1}{2}}$, and $2(3ax)^{\frac{1}{2}}$ by $7(3ay)^{\frac{1}{2}}$.

Q. Square $4x^{\frac{1}{2}}+5y^{\frac{1}{2}}$, and multiply $2x^{\frac{1}{2}}+5$ by $x^{\frac{1}{2}}-2$.

R. Find the product of $+2(abc)^{\frac{1}{3}}$, $-3(a^2c)^{\frac{1}{3}}$, and $-4(b^2c)^{\frac{1}{3}}$.

S. Divide $20(15)^{\frac{1}{2}}$ by $4(5)^{\frac{1}{2}}$, $45(21)^{\frac{1}{2}}$ by 9, and $45(21)^{\frac{1}{3}}$ by $3^{\frac{1}{3}}$.

T. Divide $12(6xy)^{\frac{1}{2}}$ by $4(3y)^{\frac{1}{2}}$, and $3(x^4y)^{\frac{1}{3}}$ by $5(xy)^{\frac{1}{3}}$.

2. Divide $6+x^{\frac{1}{2}}-2x$ by $2-x^{\frac{1}{2}}$. Ans. $3+2x^{\frac{1}{2}}$.

3. Divide $6y-2y^{\frac{1}{2}}-20$ by $3y^{\frac{1}{2}}+5$. Ans. $2y^{\frac{1}{2}}-4$.

U. Divide $a-25$ by $a^{\frac{1}{2}}-5$.

V. Divide $9+6a^{\frac{1}{2}}+a$ by $3+a^{\frac{1}{2}}$.

W. Extract the square root of $225-30y^{\frac{1}{2}}+y$.

X. Express $\{4x-12(xy)^{\frac{1}{2}}+9y\}^{\frac{1}{2}}$ in its simplest form.

## THEOREM I.

*Both Terms of a fractional Exponent may be multiplied, or both divided, by the same Number, without altering the Value of the Expression in which it occurs.* Thus $a^{\frac{2}{3}} \doteq a^{\frac{4}{6}}$.

DEM. It is not enough, to show that $\frac{2}{3} \doteq \frac{4}{6}$, since it has not yet been proved that exponents in this form are really fractions, although read as such and called so. But the theorem may be proved thus :—

It has been shown, in Ch. XXVIII. Theorem, that $\sqrt[mn]{a} \doteq \sqrt[m]{\sqrt[n]{a}}$.

But, by Def. 1, $a^{\frac{4}{6}} \doteq \sqrt[6]{a^4} = \sqrt[3]{\sqrt[2]{a^4}} = \sqrt[3]{a^2} = a^{\frac{2}{3}}$. And in like manner it may be proved that $a^{\frac{mp}{np}} \doteq a^{\frac{m}{n}}$.

EXERCISE A. Express $a^{\frac{18}{36}}$ and $b^{\frac{28}{35}}$ in their simplest forms.

## THEOREM II.

*All the Rules given, in the preceding Part of this Volume, for integral Exponents, are applicable to fractional Exponents, treated as Fractions.*

Thus $a^{\frac{m}{n}} \times a^{\frac{p}{q}} \doteq a^{\frac{m}{n}+\frac{p}{q}}$, $a^{\frac{m}{n}} \div a^{\frac{p}{q}} = a^{\frac{m}{n}-\frac{p}{q}}$, and $\left(a^{\frac{m}{n}}\right)^{\frac{p}{q}} = a^{\frac{m}{n}\times\frac{p}{q}}$.

DEM. $a^{\frac{1}{3}} \times a^{\frac{1}{4}} \doteq a^{\frac{1}{3}+\frac{1}{4}}$, because $a^{\frac{1}{3}} \doteq a^{\frac{4}{12}}$ (Th. I.) $= \sqrt[12]{a^4}$ (Def. 1), and $a^{\frac{1}{4}} \doteq a^{\frac{3}{12}} = \sqrt[12]{a^3}$, and $\sqrt[12]{a^4} \times \sqrt[12]{a^3} \doteq \sqrt[12]{(a^4 \times a^3)}$ (Ch. XVI. Pr. II.) $= \sqrt[12]{a^7} = a^{\frac{7}{12}}$, which is exactly the result we should have attained by adding the exponents $\frac{1}{3}$ and $\frac{1}{4}$ as fractions.

In like manner it may be proved that $a^{\frac{2}{3}} \times a^{\frac{3}{4}} = a^{\frac{2}{3}+\frac{3}{4}}$, and, in general, that $a^{\frac{m}{n}} \times a^{\frac{p}{q}} \doteq a^{\frac{m}{n}+\frac{p}{q}}$.

Again, since $a^{\frac{m}{n}-\frac{p}{q}} \times a^{\frac{p}{q}} \doteq a^{\frac{m}{n}}$, as has just been proved, it follows that $a^{\frac{m}{n}} \div a^{\frac{p}{q}} \doteq a^{\frac{m}{n}-\frac{p}{q}}$.

Lastly, $\left(a^{\frac{m}{n}}\right)^3 \doteq a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} = a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}}$ (as proved above) $= a^{3\frac{m}{n}}$, and, generally, as may be shown in the same manner, $\left(a^{\frac{m}{n}}\right)^p \doteq a^{\frac{mp}{n}}$. Then $\left(a^{\frac{m}{n}}\right)^{\frac{p}{q}} \doteq \sqrt[q]{\left(a^{\frac{m}{n}}\right)^p} = \sqrt[q]{a^{\frac{mp}{n}}}$ $= \sqrt[nq]{a^{mp}}$ (as proved in the Theorem of Ch. XXVIII.) $= a^{\frac{mp}{nq}}$ (Def. 1.)

EXERCISES. What are the simplest forms of expression for the following products, powers, and quotients.

1. $5^{\frac{1}{3}} \times 5^{\frac{2}{3}}$, and $a^{\frac{3}{8}} \times a^{\frac{1}{8}}$. Ans. 5 and $a^{\frac{1}{2}}$.

A. $c^{\frac{3}{7}} \times c^{\frac{4}{7}}$, $d^{\frac{5}{8}} \times d^{\frac{3}{8}}$, and $e^{\frac{3}{5}} \times e^{\frac{4}{5}}$.

2. $5^{\frac{1}{3}} \times 5^{\frac{1}{4}}$, $x \times x^{\frac{1}{2}}$, $a^{\frac{2}{3}} \times a^{\frac{1}{4}}$, and $(b^{\frac{3}{4}})^5$.
Ans. $78125^{\frac{1}{12}}$, $x^{\frac{3}{2}}$, $a^{\frac{11}{12}}$, and $b^{\frac{15}{4}}$.

B. $(4^{\frac{1}{3}})^3$, $(4^{\frac{2}{3}})^2$, and $(4^{\frac{2}{3}})^{12}$.

C. $b^{\frac{3}{4}} \times b^{\frac{1}{4}}$, $(2^{\frac{2}{3}})^5$, and $(a^{\frac{2}{3}})^{\frac{3}{2}}$.

D. $(x^{\frac{5}{6}})^2 \times x^{\frac{3}{4}}$, $(y^{\frac{2}{3}})^{\frac{3}{4}}$, $\sqrt[3]{w^{\frac{5}{4}}}$, and $(z^8)^{\frac{3}{4}}$.

E. $a^{\frac{3}{4}} \div a^{\frac{1}{4}}$, $b^{\frac{3}{2}} \div b^{\frac{2}{3}}$, $x^{\frac{1}{3}} \div x$, $y \div y^{\frac{5}{6}}$, and $z \div z^{\frac{m}{n}}$.

F. $c^3 \div c^{\frac{3}{4}}$, $(x^{\frac{3}{4}})^2 \div x^{\frac{5}{4}}$, $(y^{\frac{2}{3}})^{\frac{3}{5}} \times (y^{\frac{3}{5}})^{\frac{2}{3}} \div y^{\frac{2}{5}}$, and $z^{\frac{1}{n}} \div z^{\frac{1}{m}}$.

3. $(\frac{1}{2}a^{\frac{3}{5}} + \frac{2}{3}a^{\frac{2}{5}}b^{\frac{1}{5}} - \frac{3}{4}a^{\frac{1}{5}}b^{\frac{2}{5}} - \frac{4}{5}b^{\frac{3}{5}}) \times (6a^{\frac{2}{5}}b^{\frac{2}{5}})$.
Ans. $3ab^{\frac{2}{5}} + 4a^{\frac{4}{5}}b^{\frac{3}{5}} - \frac{9}{2}a^{\frac{3}{5}}b^{\frac{4}{5}} - \frac{24}{5}a^{\frac{2}{5}}b$.

G. $(2a - a^{\frac{2}{3}}b^{\frac{1}{3}} + \frac{2}{3}a^{\frac{1}{3}}b^{\frac{2}{3}} - \frac{3}{4}b) \times 12a^{\frac{1}{3}}b^{\frac{2}{3}}$.

4. $(a^{\frac{1}{2}} - b^{\frac{1}{3}})^3$. Ans. $a^{\frac{3}{2}} - 3ab^{\frac{1}{3}} + 3a^{\frac{1}{2}}b^{\frac{2}{3}} - b$.

H. $(a^{\frac{2}{3}} - a^{\frac{1}{3}} + 1)^2$.

5. $(x^{\frac{2}{3}} + 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}) \times x^{\frac{1}{3}} - y^{\frac{1}{3}}$. Ans. $x + x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}y^{\frac{2}{3}} - y$.

K. $(9a^{\frac{2}{3}} + 6a^{\frac{1}{3}}b^{\frac{1}{3}} + 4b^{\frac{2}{3}}) \times (3a^{\frac{1}{3}} - 2b^{\frac{1}{3}})$.

6. $(a^{\frac{1}{2}} + 6a^{\frac{1}{4}}b^{\frac{1}{4}} + 9b^{\frac{1}{2}}) \times (a^{\frac{1}{2}} - 6a^{\frac{1}{4}}b^{\frac{1}{4}} + 9b^{\frac{1}{2}})$.
Ans. $a - 18a^{\frac{1}{2}}b^{\frac{1}{2}} + 81b$.

L. $(a^{\frac{1}{6}} - 3a^{\frac{1}{3}} + 9a^{\frac{1}{2}}) \times (1 + 3a^{\frac{1}{6}})$.

M. $(4ab^{\frac{1}{5}} - 3a^{\frac{4}{5}}b^{\frac{2}{5}} + a^{\frac{3}{5}}b^{\frac{3}{5}} - \frac{1}{2}a^{\frac{2}{5}}b^{\frac{4}{5}}) \div 2a^{\frac{2}{5}}b^{\frac{1}{5}}$.

7. $(z - 3125) \div (z^{\frac{1}{5}} - 5)$.
Ans. $z^{\frac{4}{5}} + 5z^{\frac{3}{5}} + 25z^{\frac{2}{5}} + 125z^{\frac{1}{5}} + 625$.

N. $(64a^2 - 729) \div (4a^{\frac{2}{3}} - 9)$.

8. $(x - 5x^{\frac{4}{5}}y^{\frac{1}{4}} + 10x^{\frac{3}{5}}y^{\frac{1}{2}} - 10x^{\frac{2}{5}}y^{\frac{3}{4}} + 5x^{\frac{1}{5}}y - y^{\frac{5}{4}}) \div (x^{\frac{2}{5}} - 2x^{\frac{1}{5}}y^{\frac{1}{4}} + y^{\frac{1}{2}})$. Ans. $x^{\frac{3}{5}} - 3x^{\frac{2}{5}}y^{\frac{1}{4}} + 3x^{\frac{1}{5}}y^{\frac{1}{2}} - y^{\frac{3}{4}}$.

O. $(25a^{\frac{4}{3}} + 50a^{\frac{2}{3}} - 100a^{\frac{1}{3}} + 200) \div (5a^{\frac{2}{3}} - 10a^{\frac{1}{3}} + 10)$.

9. $(4a^{\frac{6}{5}} - 12a + 9a^{\frac{4}{5}})^{\frac{1}{2}}$. Ans. $2a^{\frac{3}{5}} - 3a^{\frac{2}{5}}$.

P. $(a^{\frac{4}{3}} - 2ab^{\frac{1}{3}} + 3a^{\frac{2}{3}}b^{\frac{2}{3}} - 2a^{\frac{1}{3}}b + b^{\frac{4}{3}})^{\frac{1}{2}}$.

10. $(x+y)^{\frac{1}{2}}$. Ans. $x^{\frac{1}{2}} + \frac{y}{2x^{\frac{1}{2}}} - \frac{1y^2}{2.4x^{\frac{3}{2}}} + \frac{1.3y^3}{2.4.6x^{\frac{5}{2}}} -$ &c.

Resolve the following quadratic and cubic equations:—

11. $x^{\frac{1}{3}} - x^{\frac{1}{6}} = 20$. Ans. $x = 15625$.

Q. $y^{\frac{2}{3}} - 2y^{\frac{1}{3}} - 35 = 0$.

12. $4z^{\frac{3}{4}} - z^{\frac{3}{8}} - 201 = 47$. Ans. $z = 256$ or $(-\frac{31}{4})^{\frac{8}{3}}$.

R. $\frac{1}{2}x - \frac{1}{4}x^{\frac{1}{2}} - 50 = 58\frac{3}{4}$.

13. $\begin{cases} x^{\frac{4}{3}} - y^{\frac{4}{3}} = 671, \\ x^{\frac{2}{3}} - y^{\frac{2}{3}} = 11. \end{cases}$ Ans. $\begin{cases} x = \pm 216, \\ y = \pm 125. \end{cases}$

S. $\left\{\begin{array}{l} x^{\frac{3}{4}} + y^{\frac{3}{2}} = \phantom{2}35, \\ x^{\frac{3}{4}} \times y^{\frac{3}{2}} = 216. \end{array}\right\}$

14. $\left\{\begin{array}{l} y^{\frac{2}{5}} + y^{\frac{1}{5}}z^{\frac{1}{5}} = 21, \\ y^{\frac{1}{5}}z^{\frac{1}{5}} + z^{\frac{2}{5}} = 28. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} y = 243, \\ z = 1024. \end{array}\right.$

T. $\left\{\begin{array}{l} x^{\frac{2}{3}} + y^{\frac{2}{3}} = 521, \\ x^{\frac{1}{3}} + y^{\frac{1}{3}} = \phantom{5}31. \end{array}\right\}$

15. $x^{\frac{7}{3}} + \frac{41\sqrt[3]{x}}{x} = \frac{97}{\sqrt[3]{x^2}} + x^{\frac{5}{6}}$. Ans. $x = 4$ or $(-7)^{\frac{2}{3}}$.

U. $y - 6y^{\frac{1}{2}} = 40$.

16. $x^{\frac{3}{2}} - 12x + 48x^{\frac{1}{2}} = 280$. Ans. $x = 100$.

V. $x^{\frac{3}{4}} - 3x^{\frac{1}{2}} + 6x^{\frac{1}{4}} = 8$.

**Def.** 2. Any quantity with the exponent *zero*, or 0, means 1. Thus $1^0$, $5^0$, $(\frac{1}{4})^0$, and $100^0$ are severally $= 1$, and, generally, $a^0 \doteq 1$.

That this is consistent with analogy will be obvious; for, if we continue the series $a^4$, $a^3$, $a^2$, $a^1$, one term further, by diminishing the last exponent by another unit, the next term becomes $a^0$, and, if we continue the series by dividing by $a$, the next term is found to be 1.

**Cor.** Hence $a^n \div a^n \doteq a^0$, for any quantity divided by itself gives 1, which, by the def., $\doteq a^0$.

**Def.** 3. A quantity with a *negative exponent* means the reciprocal of the same quantity with the same exponent in a positive form. Thus $5^{-2}$ means $\frac{1}{5^2}$, and $a^{-n} \doteq \frac{1}{a^n}$.

## Theorem III.

*When a lower Power of any Quantity is to be divided by a higher Power of the same Quantity, the Quotient may be expressed by that Quantity, with a negative Exponent equal to the Difference of the two given Exponents.*

Thus $5^3 \div 5^7 \doteq 5^{-4}$, and, generally, if $m$ be greater than $n$, $a^n \div a^m \doteq a^{-(m-n)}$.

**Dem.** In the dem. of Pr. III. of Ch. XI., it is shown that the fraction $\frac{b}{c} \doteq b \div c$. $\therefore 5^3 \div 5^7 \doteq \frac{5^3}{5^7} = \frac{1}{5^4}$ (Ch. XI.

Th.) $= 5^{-4}$ (Def. 3); and, in like manner, $a^n \div a^m = \frac{a^n}{a^m}$ $= \frac{1}{a^{m-n}} = a^{-(m-n)}$.

**Cor. 1.** The same demonstration is alike applicable to fractional exponents; for $a^{\frac{2}{7}} \div a^{\frac{5}{7}} = \frac{a^{\frac{2}{7}}}{a^{\frac{5}{7}}} = \frac{1}{a^{\frac{3}{7}}} = a^{-\frac{3}{7}}$, and, generally, by a similar proof, if $\frac{m}{n}$ be greater than $\frac{p}{q}$, $a^{\frac{p}{q}} \div a^{\frac{m}{n}}$ $\doteq a^{-\left(\frac{m}{n}-\frac{p}{q}\right)}$.

**Cor. 2.** Hence the rule becomes universal, that, *to divide one power by another power of the same quantity, whether a higher power or a lower, or equal, we may subtract the exponent of the divisor from that of the dividend.*

**Exercise A.** Divide $2^5$ by $2^{12}$, $a^{\frac{1}{2}}$ by $a^{\frac{1}{3}}$, and $5^{\frac{2}{3}}$ by $5^{\frac{3}{4}}$.

**Cor. 3.** In this manner any fraction may be written in an integral *form*, by writing the denominator as a factor of the numerator, at the same time changing its sign. The same remark is applicable to a quotient.

Thus $\frac{a^2}{b^3}$ may be written $a^2b^{-3}$, $\frac{x^2}{y} \doteq x^2y^{-1}$, $\frac{1}{3} \doteq 3^{-1}$, and $2 \div 3$ may be written $2^1.3^{-1}$.

If the divisor or denominator is a compound quantity, we must take it as a whole, and not change the signs of the exponents of its separate terms. Thus $\frac{1}{a^2+x^2}$ is not equivalent to $a^{-2}+x^{-2}$, but to $(a^2+x^2)^{-1}$.

**Exercise B.** Express the following fractions and quotients in an integral form; viz. $\frac{2^m}{3^n}$, $\frac{a^5}{b^2}$, $a \div b^3$, $a^3 \div b$, $\frac{5}{9}$, $\frac{(a+b)^2}{(a-b)^2}$, $\frac{a^2+b^2}{a^2-b^2}$, and $\frac{1}{a^2} - \frac{1}{b^2}$.

**Cor. 4.** In like manner any factor with a negative exponent, in a numerator, or in any expression with an integral form, may be written as a denominator with a positive exponent. Thus $a^m b^{-m}$ may be written $\frac{a^m}{b^m}$ or $\left(\frac{a}{b}\right)^m$.

**Exercise C.** Change the following expressions into others equivalent to them with only positive exponents;— $5^2.4^{-3}$, $a^{-m}b^n$, $x^{-5}$, $xy^{-1}$, $(xy)^{-1}$, $x^{-1}y$, and $x^{-3}-3x^{-2}y^{-1}+3x^{-1}y^{-2}-y^{-3}$.

COR. 5. A quantity with a negative exponent, in a denominator, may be written with a positive exponent in a numerator.

Thus $\dfrac{a^m}{b^{-n}} \doteq a^m b^n$; for $\dfrac{a^m}{b^{-n}} \doteq a^m \div \dfrac{1}{b^n} = \dfrac{a^m}{1} \times \dfrac{b^n}{1} = a^m b^n$.

EXERCISE D. Express the following quantities with positive exponents only;—$\dfrac{a^3}{x^{-2}}$, $\dfrac{1}{c^{-5}}$, $\dfrac{ab^2x^{-3}}{c^2z^{-4}}$, $\dfrac{3^{-m}}{5^{-n}}$, $\dfrac{ax^{-1}}{by^{-1}}$, and $\dfrac{(ax)^{-1}}{(by)^{-1}}$.

## THEOREM IV.

*All the Rules regarding Quantities with positive Exponents may be applied to Quantities having negative Exponents, the negative Exponents being treated as negative Quantities.*

Thus $a^m \times a^{-n} \doteq a^{m-n}$, $a^{-m} \times a^{-n} \doteq a^{-(m+n)}$, $a^{-m} \div a^n = a^{-m-n}$, $a^m \div a^{-n} = a^{m+n}$, $a^{-n} \div a^{-m} \doteq a^{m-n}$, $(a^{-m})^n \doteq a^{-mn}$, $(a^m)^{-n} = a^{-mn}$, $(a^{-m})^{-n} \doteq a^{mn}$, whether $m$ and $n$ be integers or fractions.

DEM. $a^m \times a^{-n} \doteq a^m \times \dfrac{1}{a^n} = \dfrac{a^m}{a^n} = a^{m-n}$ (Th. III. Cor. 2.)

$$a^{-m} \times a^{-n} \doteq \frac{1}{a^m} \times \frac{1}{a^n} = \frac{1}{a^{m+n}} = a^{-(m+n)}.$$

$$a^{-m} \div a^n \doteq \frac{1}{a^m} \times \frac{1}{a^n} = \frac{1}{a^{m+n}} = a^{-m-n}.$$

$a^m \div a^{-n} \doteq a^m \times a^n$ (Th. III. Cor. 5) $= a^{m+n}$.

$$a^{-n} \div a^{-m} \doteq a^{-n} \times a^m = a^{m-n}.$$

$$(a^{-m})^n \doteq \left(\frac{1}{a^m}\right)^n = \frac{1}{a^{mn}} = a^{-mn}.$$

$$(a^m)^{-n} = \frac{1}{(a^m)^n} = \frac{1}{a^{mn}} = a^{-mn}.$$

$$a^{-m})^{-n} \doteq \left(\frac{1}{a^m}\right)^{-n} = 1 \div \left(\frac{1}{a^m}\right)^n = 1 \div \frac{1}{a^{mn}} = 1 \times \frac{a^{mn}}{1} = a^{mn}.$$

EXERCISES. Express the following products, &c. in their simplest forms.

A. $2^7\times2^{-3}$, $a^{-2}\times a^5$, $a\times a^{-1}$, $a^{-2}\times a^{-3}$, and $b^{\frac{1}{2}}\times b^{-\frac{3}{2}}$.

B. $3^{-3}\div3^{-5}$, $a^4\div a^{-3}$, $a^{-4}\div a^3$, $b^{\frac{1}{2}}\div b^{-\frac{1}{2}}$, $a^{-1}x^{\frac{1}{3}}\div ax^{-\frac{2}{3}}$.

C. $(a^{-3})^2$, $(b^4)^{-3}$, $(c^{-5})^{-4}$, $(c^2d^{-4})^{\frac{1}{2}}$, and $(x^{\frac{1}{3}}y^{\frac{1}{2}})^{-6}$.

D. $(a-2a^{\frac{1}{2}}+3-2a^{-\frac{1}{2}}+a^{-1})\times a^{-\frac{3}{2}}$.

E. $(a-2a^{\frac{1}{2}}+3-2a^{-\frac{1}{2}}+a^{-1})\div a^{-\frac{3}{2}}$.

F. $(a^{-2}-2a^{-1}b^{-1}+b^{-2})^2$.

1. $(3x^{-\frac{1}{2}}y^{\frac{3}{2}}+2x^{\frac{3}{2}}y^{-\frac{1}{2}})\times(3x^{-\frac{1}{2}}y^{\frac{3}{2}}-2x^{\frac{3}{2}}y^{-\frac{1}{2}})$.
Ans. $9x^{-1}y^3-4x^3y^{-1}$.

G. $(x^2+x^{\frac{3}{2}}y^{\frac{1}{2}}+xy+x^{\frac{1}{2}}y^{\frac{3}{2}}+y^2)\times(x^{-\frac{3}{2}}y^{-2}-x^{-2}y^{-\frac{3}{2}})$.

2. $(x^2+x^{\frac{3}{2}}y^{\frac{1}{2}}+xy+x^{\frac{1}{2}}y^{\frac{3}{2}}-4y^2)\div(x^{-\frac{3}{2}}y^{-2}-x^{-2}y^{-\frac{3}{2}})$.
Ans. $x^{\frac{7}{2}}y^2+2x^3y^{\frac{5}{2}}+3x^{\frac{5}{2}}y^3+4x^2y^{\frac{7}{2}}$.

H. $\sqrt{}(x^{-2}-6x^{-1}y^{-1}+9y^{-2})$.

3. $(a^{-6}-4a^{-5}-2a^{-4}+12a^{-3}+9a^{-2})^{\frac{1}{2}}$.
Ans. $a^{-3}-2a^{-2}-3a^{-1}$.

K. $\sqrt{}(49a^{-4}-70a^{-3}b^{-1}+67a^{-2}b^{-2}-30a^{-1}b^{-3}+9b^{-4})$.

Resolve the following equations :—

4. $x^{-2}-8x^{-1}=33$. Ans. $x=\frac{1}{11}$ or $-\frac{1}{3}$.

L. $x^{-2}+12x^{-1}=160$.

5. $7y^{-6}-63y^{-3}+56=0$. Ans. $y=1$ or $\frac{1}{2}$.

M. $z^{-2}(4z^{-2}-5)=-1$.

6. $\left\{\begin{array}{l} x^{-2}-y^{-2}=9\times20^{-2}, \\ x^{-1}-y^{-1}=20^{-1}. \end{array}\right\}$ Ans. $\left\{\begin{array}{l} x=4. \\ y=5. \end{array}\right.$

N. $\left\{\begin{array}{l} x^{-2}+y^{-2}=13\times6^{-2}, \\ x^{-1}-y^{-1}=\ \ 6^{-1}. \end{array}\right\}$

7. $10x^{-\frac{3}{2}}+\frac{1}{2}x^{\frac{1}{2}}-6x^{-\frac{1}{2}}=0$. Ans. $x=10$ or $2$.

O. $z^{-3}+\dfrac{1}{z\sqrt{z}}=2$.

# CHAPTER XXXII.

## INDETERMINATE COEFFICIENTS.

### THEOREM I.

*In any Series of the Form* $ax+bx^2+cx^3+$ &c., $x$ *may be taken so small that the sum of the whole Series* (s) *shall be less than any given positive Quantity* (q).

DEM. Of all the coefficients, $a$, $b$, $c$, &c., whether positive or negative, let $m$ represent the greatest, or, more correctly, that which is not less than any other.*

Then $s$ is evidently not $> mx + mx^2 + mx^3 + \&c.$, or $mx \times (1 + x + x^2 + \&c.)$ But $\frac{1}{1-x} \doteq 1 + x + x^2 + \&c.$ continued *ad infinitum*, when $x$ is $< 1$. $\therefore$, when $x$ is $< 1$, $s$ is not $> mx \times \frac{1}{1-x}$, or $\frac{mx}{1-x}$ is not $< s$.

If we take $x$ so small that it is $< \frac{q}{m+q}$, and also† $< 1$, $(m+q)x$ is $< q$, $mx + qx$ is $< q$, $mx$ is $< q - qx$, or $mx$ is $< q(1-x)$. $\therefore \frac{mx}{1-x}$ is $< q$; and we have before proved that the same fraction is not $< s$. $\therefore s$ is $< q$.

NOTE. The preceding demonstration holds good, it will be observed, whether the series is finite or infinite, and whether the coefficients are positive or negative or partly positive and partly negative.

COR. Hence, in any series of the form $a + bx + cx^2$ &c., a value may be found for $x$, such that any one term whose coefficient is a positive number, shall be greater than the sum of all the terms that follow. For, if $nx^m$ be that term, then, by dividing all the subsequent terms by $x^m$, we have, of these, a series, $px + qx^2 + \&c.$, in which, by the theorem, $x$ may be taken of so small a value that all the terms together shall be less than $n$. That is, $n$ is $> px + qx^2 + \&c.$, and, consequently, $nx^m > px^{m+1} + qx^{m+2} + \&c.$

### THEOREM II.

*If any Series of the Form* $A + Bx + Cx^2 + \&c.$ *be always equal to another Series of the same Form, as* $a + bx + cx^2 + \&c.$, *whatever be the Value of* $x$, *then will the Coefficients of the respective Powers of* $x$, *in the one Series, be severally equal to the Coefficients of the same Powers in the other Series.*

That is, $A$ will be $= a$, $B = b$, $C = c$, &c.

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* In such expressions it must be distinctly understood whether we mean the greatest *numerically* or the greatest *algebraically*,—that is, disregarding or including the signs. If the quantities are all positive, the two expressions are synonymous; but if some or all of the quantities are negative the greatest numerically may be the least algebraically. Thus the coefficient of $x$ in the quantity $-5x$ is numerically greater than that of $x$ in the quantity $+3x$, but algebraically less, since 5 is $> 3$, but $-5$ is less than $+3$. In like manner $-5x$ is really, or algebraically less than $-3x$, but its coefficient is numerically greater. In this chapter the words *greater* and *less* are used throughout in their algebraical import.

† It will not be *necessarily* less than 1 in consequence of being less than $q \div (m+q)$ unless $m$ is a positive number.

This is called *equating the coefficients of like powers of* x.

Dem. When $x = 0$, all the terms of each series vanish except the first, and we have $A = a$.

Taking these equals from each side of the original equation, there remains

$$Bx + Cx^2 + \&c. = bx + cx^2 + \&c.$$

and dividing by $x$,

$$B + Cx + \&c. = b + cx + \&c.$$

Consequently, as before, $B = b$.

In the same manner we prove $C = c$, $D = d$, and so on.

But if the words "whatever be the value of $x$," in the enunciation, be understood to signify "whatever finite value be given to $x$, excluding 0 and infinity," then we must have recourse to a different mode of demonstration, proceeding thus,—

If $A$ is not $= a$, one of them must be greater than the other. Let $A$ be the greater. Then $A - a$ is a real positive quantity.

Now, since $A + Bx + Cx^2 + \&c. = a + bx + cx^2 + \&c.$, we have

$$A - a = (b - B)x + (c - C)x^2 + \&c.$$

Then, by Th. I., such a value may be given to $x$ that the whole of the second member of the equation shall be less than any given positive quantity, and consequently less than $A - a$. But it is also equal to $A - a$; which is absurd. Therefore $A$ is not $> a$, and in the same manner it may be shown that $A$ is not $< a$. $\therefore A = a$.

$$\therefore Bx + Cx^2 + \&c = bx + cx^2 + \&c.$$

$$\text{and } B + Cx + \&c. = b + cx + \&c.$$

From this we prove as before that $B = b$; then that $C = c$, and so on.

Cor. If $A + Bx + Cx^2 + \&c. = 0$, under all values of $x$, then $A = 0$, $B = 0$, $C = 0$, &c.

Example. Let it be required to divide 1 by $1 + x$.

Assume $1 \div (1 + x) = A + Bx + Cx^2 + Dx^3 + \&c.$

Multiply both sides by $1 + x$, and we have

$$1 = A + (A + B)x + (B + C)x^2 + \&c.$$

Then, since the left-hand member of the equation may be expressed as $1 + 0x + 0x^2 + \&c.$, we have $A = 1$, $A + B = 0$, $B + C = 0$, $C + D = 0$, &c.

$\therefore A \doteq +1$, $B = -1$, $C = +1$, $D = -1$, and so on; and

$$1 \div (1 + x) \doteq 1 - x + x^2 - x^3 + \&c.$$

**Note.** The assumption with which we commenced may be considered unwarrantable; but it amounted to no more than that the quotient will be of the *form* supposed; and when, in any case, the assumed form is impossible, the equations of the coefficients will either be contradictory, or else consistent only for infinite values of the coefficients.

**Example 2.** Find the square root of $1 + x^2$.

Assume $\sqrt{(1 + x^2)} = A + Bx + Cx^2 + \&c.$

Squaring both sides, $1 + x \doteq A^2 + 2ABx + (2AC + B^2)x^2 + (2AD + 2BC)x^3 + (2AE + 2BD + C^2)x^4 + \&c.$

$$\therefore A^2 \doteq 1, \text{ and } A = 1;$$
$$2AB \doteq 1,\ 2B = 1, \text{ and } B = \tfrac{1}{2};$$
$$2AC + B^2 \doteq 0,\ 2C = -\tfrac{1}{4}, \text{ and } C = -\tfrac{1}{8};$$
$$2AD + 2BC \doteq 0,\ AD + BC = 0,\ D - \tfrac{1}{16} = 0,\ D = +\tfrac{1}{16};$$
$$2AE + 2BD + C^2 \doteq 0,\ 2E + \tfrac{1}{16} + \tfrac{1}{64} = 0,\ E = -\tfrac{5}{128}.$$
$$\therefore \sqrt{(1 + x)} \doteq 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 + \&c.$$

If we multiply both terms of each of the last two fractions by 3, and resolve the denominators into the proper factors, the series assumes order, becoming

$$1 + \tfrac{1}{2}x - \frac{1}{2.4}x^2 + \frac{1.3}{2.4.6}x^3 - \frac{1.3.5}{2.4.6.8}x^4 + \&c.$$

**Exercise 1.** Divide 1 by $1 - x$, by the method of indeterminate coefficients. Ans. $1 + x + x^2 + x^3 + \&c.$

A. Divide $1 + 3x$ by $1 - 3x$.

2. Expand $\dfrac{1}{1 + x + x^2}$ into a series.

Ans. $1 - x + x^3 - x^4 + x^6 - x^7 + \&c.$,
or $(1 - x)(1 + x^3 + x^6 + x^9 + \&c.)$

B. Expand $\dfrac{a}{1 - x - x^2}$ into a series.

3. Express the square root of $a^2 - x^2$ in the form of a series.

Ans. $a - \frac{1}{2}.\dfrac{x^2}{a} - \dfrac{1}{2.4}.\dfrac{x^4}{a^3} - \dfrac{1.3}{2.4.6}.\dfrac{x^6}{a^5} - \dfrac{1.3.5}{2.4.6.8}.\dfrac{x^8}{a^7} - \&c.$

C. Develope $\sqrt[3]{(1 + a)}$ into a series.

# CHAPTER XXXIII.

## VARIATIONS AND COMBINATIONS.

The different orders in which any objects can be arranged in equal groups, of a given number each, are called their *Variations.*

Thus the variations of the letters *abc*, taken singly, are *a*, *b*, *c*; their variations, taken two and two together, are *ab*, *ba*, *ac*, *ca*, *bc*, *cb*; their variations taken all together are *abc*, *acb*, *bac*, *bca*, *cab*, *cba*.

When the objects are taken all together, the variations are called *Permutations.* But the distinction of the words *variation* and *permutation* is not always observed, these words being often used indiscriminately, as also *alternations* and *changes.*

The *Combinations* of any objects are the different ways in which they can be grouped together without reference to their places in the group. Thus *ab* and *ba*, although different variations, are the same combination.

The single letters, *V*, *P* and *C*, are employed to denote variations, permutations and combinations. When a small figure is attached to the letter, on the left and above, it denotes the whole number of objects. Another figure on the right of the *V* or *C* and below it shows the number taken in each group. Thus ${}^{10}V_3$ means the number of variations of ten objects taken three by three; ${}^{n}C_r$ means the number of combinations of *n* objects taken in groups containing *r* each; and ${}^{n}V_n \doteq {}^{n}P$.

### THEOREM I.

*The Number of Variations of* n *Things, taken* r *together, is* n (n − 1) (n − 2) (n − 3) &c. *continued to* r *terms,—or*

$$ {}^{n}V_r \doteq n(n-1)(n-2)\ldots\ldots\ldots\ldots(n-\overline{r-1}). $$

DEM. In *n* things, *a*, *b*, *c*, *d*, &c. taken two together, *a* may be placed before each of the others, as *ab*, *ac*, *ad*, &c. forming $n-1$ variations, since there are $n-1$ letters which can be placed after *a*. For the same reason there are $n-1$ variations in which *b* stands first, and so on. Therefore there are altogether $n$ times $(n-1)$ variations of this kind, or ${}^nV_2 \doteq n(n-1)$.

Again, of $n-1$ things $b$, $c$, $d$, &c., taken two together, there are $(n-1)(n-2)$ variations, $n-1$ taking the place of $n$ in the preceding paragraph. Prefixing $a$ to each of these variations, there are $(n-1)(n-2)$ variations, taken three and three, in which $a$ stands first; and the same may be said of $c$, $d$, &c. Therefore there are, upon the whole, $n(n-1)(n-2)$ variations of $n$ objects in threes, or ${}^nV_3 \doteq n(n-1)(n-2)$.

By following out the same method, we find ${}^nV_4 = n(n-1)(n-2)(n-3)$, and so on, the number of factors being $r$, and the last factor $n-(r-1)$, or $n-r+1$.

**Example.** Required the number of variations of 100 objects taken in fives.

$${}^{100}V_5 \doteq 100 \times 99 \times 98 \times 97 \times 96 = 9{,}034{,}502{,}400.$$

**Exercise** 1. How many variations can be made of twelve things taken six together? Ans. 665,280.

A. What is the value of ${}^{20}V_{10}$?

2. In a school consisting of 13 boys, one of them engaged to his companion that, for the sum of one guinea, paid down, he would present him with a farthing for every time he could turn out a group of five different boys, or of the same five in a different order. Did the former gain or lose by the bargain, and how much?

Ans. He lost £159, 16s. 6d.

B. Compute the numerical values of ${}^{53}V_7$, and ${}^{10}V_9$.

3. How many changes can be rung with 6 bells out of 7? Ans. 5040.

C. How many numbers can be written of 3 figures each, a cipher being admitted, but without taking the same figure twice in the same number, and taking such an expression as 058 to mean 58?

## THEOREM II.

*The Number of Permutations of* n *Things is* n(n – 1) (n – 2)............3.2.1.

**Dem.** Since the permutations of $n$ objects are the same as their variations when taken all together, the rule for variations will hold for permutations if for $r$ we substitute $n$, the last term becoming $n-(n-1)$, or simply 1.

Example. In how many different ways can the letters of the word Edinburgh be arranged?

$$^9P \doteq 9.8.7.6.5.4.3.2 = 362,880.$$

Exercise 1. How many changes can be rung upon seven bells, ringing the whole peal at each change?

Ans. 5040.

A. In how many different orders can a company of eleven soldiers walk in single file?

2. How many different arrangements can be made of the letters in the word *mutiny*? Ans. 720.

B. How many different arrangements can be made in a bow of the seven prismatic colours?

3. What is the value of ${}^{12}P$? Ans. 479,001,600.

C. What is the value of ${}^{15}P$?

Note 1. On comparing Ex. 1 of this theorem with Ex. 3 of the preceding, it will be seen that ${}^7P$, or ${}^7V_7 \doteq {}^7V_6$. The reason is obvious; for the factors which compose the latter are the same as those which compose the former with the omission of the last, which is 1. In like manner, generally, ${}^nP$, or ${}^nV_n \doteq {}^nV_{n-1}$.

Note 2. We are sometimes required to find the whole number of variations of $n$ objects, taking *any number* at a time. To obtain this we must first find the number of variations of the objects taken singly, then the number of variations when they are taken in twos, then in threes, and so on, the last being the number of permutations. All these numbers added together give the whole number of variations of all kinds. If we proceed in the order here described, each number may be obtained from the preceding by one multiplication.

Exercise 4. How many numbers, of one, two, or more figures, can be made up of the nine digits, without using the cipher or taking the same figure twice in the same number? Ans. 986,409.

Note 3. When the same thing is required, but the largest number which may be taken together is less than *the whole* number of objects, we must proceed in the same manner, stopping when we have attained the variations in groups of the highest assigned limit.

**Exercise** 5. How many numbers, of not more than five figures each, can be made with the nine digits, on the conditions prescribed in the preceding exercise?

Ans. 18,729.

### Theorem III.

*The Number of Combinations of* n *Things, taken* r *together, is*

$$\frac{n(n-1)(n-2)\ldots\ldots\ldots\ldots(n-\overline{r-1})}{1.2.3\ldots\ldots\ldots r}.$$

**Dem.** By Th. I., ${}^nV_2 \doteq n(n-1)$. But each combination of two things admits of two variations, as *ab*, *ba*. $\therefore$ the number of combinations of $n$ things taken in twos, is equal to half the number of variations, or ${}^nC_2 \doteq \frac{n(n-1)}{1.2}$.

Next, ${}^nV_3 \doteq n(n-1)(n-2)$. But each combination of three things admits of six variations, as *abc*, *acb*, *bac*, *bca*, *cab*, *cba*; for ${}^3P \doteq 3.2.1 = 6$. $\therefore {}^nC_3 \doteq \frac{n(n-1)(n-2)}{1.2.3}$.

In like manner ${}^nV_r \doteq n(n-1)(n-2)\ldots\ldots\ldots(n-\overline{r-1})$; but every combination of $r$ things admits of $1.2.3\ldots\ldots r$ variations, taken all together, since ${}^rP \doteq r(r-1)\ldots\ldots 3.2.1$.

$$\therefore {}^nC_r \doteq \frac{n(n-1)(n-2)\ldots\ldots(n-\overline{r-1})}{1.2.3\ldots\ldots r}.$$

**Cor.** 1. There will always be as many factors in the numerator as in the denominator; for, if we diminish each factor in the num$^r$ by a unit, it will become $(n-1)(n-2)(n-3)\ldots\ldots(n-r)$, the number of factors being then evidently $r$, as in the den$^r$.

**Cor.** 2. Any factor in the num$^r$ may be obtained from the corresponding factor in the den$^r$, by taking 1 from the latter and subtracting the remainder from $n$.

**Exercise** 1. How many different combinations can be formed, each of three of the seven prismatic colours?

Ans. 35.

A. A lady has, in her garden, a large quantity of flowers, of 15 varieties, which she expects to bloom about the same time. She directs her gardener to plant them in plots of 5 each, making as many plots as he can, provided that no two plots contain exactly the same flowers. How many such plots will she have?

2. How many combinations can be made of 10 things taken 3 together, and how many of 10 things taken 7 together? Ans. 120.

B. How many combinations can be formed of 12 things taken in fours, and how many of 12 taken in eights?

Cor. 3. From the two preceding exercises it will be seen that ${}^{10}C_3 \doteq {}^{10}C_7$, and that ${}^{12}C_8 \doteq {}^{12}C_4$; for, if we take the former of these instances, ${}^{10}C_3 \doteq \frac{10.9.8}{1.2.3}$, and ${}^{10}C_7 = \frac{10.9.8.7.6.5.4}{1.2.3.4.5.6.7}$ $= \frac{10.9.8}{1.2.3}$, since the factors after 8 in the num$^r$ are the same as those after 3 in the den$^r$. That this cannot but be so, will appear, if we consider that, in the expression for ${}^{10}C_7$, the fourth factor in the num$^r$ must be $10-3$ (Cor. 2), or 7, the same as the last in the den$^r$, and that the fourth in the den$^r$ is $4 = 10-(7-1)$, which is also necessarily the last in the num$^r$.

And, in general, ${}^nC_r \doteq {}^nC_{n-r}$; for, if $r$ is $> n-r$, the expression for ${}^nC_r$ may be written thus,

$$\frac{n(n-1)\ldots\ldots(r+1)\,r\,(r-1)\ldots\ldots(n-r+2)(n-r+1)}{1.2\ldots\ldots(n-r)(n-r+1)(n-r+2)\ldots\ldots(r-1)\,r},$$

in which the factor, $r+1$, of the num$^r$, corresponds in position to $n-r$ of the den$^r$, as appears from Cor. 2, and every factor following $r+1$ in the num$^r$ has its equivalent among those after $n-r$ in the den$^r$, and *vice versa*, so that the fraction, when cleared of common factors, becomes $\frac{n(n-1)\ldots.(r+1)}{1.2\ldots\ldots(n-r)} = \frac{n(n-1)\ldots.(n-\overline{n-r}+1)}{1.2\ldots\ldots(n-r)} = {}^nC_{n-r}$. But if $r$ is $< n-r$, the two merely exchange places, $n-r$ taking the place of $r$, and conversely, since $r \doteq n-(n-r)$.

The same thing may be proved in another way.—Since ${}^nC_r \doteq \frac{n(n-1)\ldots\ldots(n-r+1)}{1.2\ldots\ldots r}$, multiply both terms of the fraction by $1.2.3\ldots\ldots(n-r)$, and we have

$${}^nC_r = \frac{n(n-1)\ldots(n-r+1)\times(n-r)\ldots3.2.1}{1.2.3\ldots r\times1.2.3\ldots(n-r)} = \frac{{}^nP}{{}^rP\times{}^{n-r}P}.$$

In the first and last members of this equation, substitute

$n-r$ for $r$, and consequently $n-(n-r)$, or $r$, for $n-r$. Then ${}^{n}C_{n-r} \doteq \frac{{}^{n}P}{{}^{n-r}P \times {}^{r}P}$, which is identical with the expression for ${}^{n}C_{r}$.

This, in fact, becomes very clear from another point of view.—If, of 10 things, we take any 7, leaving 3, it is impossible to change any one of the 7 without changing one of the 3, if we must have 7 selected from the same 10. $\therefore$ we can have no more comb$^{ns}$ of sevens than of threes; for, if we had, we should have two comb$^{ns}$ of threes alike and conseq$^{y}$ two comb$^{ns}$ of sevens alike; and the same reason will apply in all other cases.

The comb$^{ns}$ with $n-r$ in each group are said to be *complementary* of those with $r$ in each.

EXERCISE 3. How many combinations can be made of 101 objects taken 98 together? Ans. 166,650.

C. Express, in numbers, ${}^{102}C_{97}$.

4. How many combinations can be formed of 10 different things, first taken in pairs, then 3 by 3, and so on, until, finally, they are taken altogether? Ans. 1013.

COR. 4. If any number of things, as $a$, $b$, $c$, be combined in every possible way, as $ab$, $ac$, $bc$, and if we add to these all the possible comb$^{ns}$ of $d$ with $a$, $b$, and $c$, as $ad$, $bd$, $cd$, we then have all the possible comb$^{ns}$ of all the things, including $d$, 2 by 2; for each thing is combined with every one of the others. In like manner, if, to all the comb$^{ns}$ of any number of quantities, as $a$, $b$, $c$, $d$, taken 3 by 3, we add all the possible comb$^{ns}$ of the same things taken 2 by 2 along with another thing, $e$, we have all the possible comb$^{ns}$ of all the things, 3 by 3; because we have, first, all the possible combinations into which $e$ does not enter, and then all those into which $e$ does enter. The same reasoning may be extended as far as we please; or, in general ${}^{n}C_{r} + {}^{n}C_{r-1} \doteq {}^{n+1}C_{r}$; for

$${}^{n}C_{r} \doteq \frac{n(n-1)\ldots.(n-r+2)(n-r+1)}{1.2\ldots.(r-1)r}$$

$${}^{n}C_{r-1} \doteq \frac{n(n-1)\ldots.(n-r+2)}{1.2\ldots.(r-1)} = \frac{n(n-1)\ldots.(n-r+2)r}{1.2\ldots.(r-1)r}$$

The sum of these is $\frac{n(n-1)\ldots.(n-r+2)}{1.2\ldots.(r-1)r} \times (n-r+1+r)$

$$= \frac{(n+1)n(n-1)\ldots.(n-r+2)}{1.2.3\ldots.r} = {}^{n+1}C_{r}.$$

# CHAPTER XXXIV.

## THE BINOMIAL THEOREM.

The primary object of the binomial theorem is to raise any binomial quantity, as $(x \pm a)$, to any given power, without going through the process of successive multiplications. The theorem is this; $(x \pm a)^n =$

$$x^n \pm \frac{n}{1}x^{n-1}a + \frac{n(n-1)}{1.2}x^{n-2}a^2 \pm \frac{n(n-1)(n-2)}{1.2.3}x^{n-3}a^3 + \&c.,$$

the upper sign in the left-hand member of the equation corresponding to the upper sign everywhere in the other, where the sign is double, and the lower to the lower.

When the object is to raise a binomial to any given power, $n$ is a positive integer. But the same theorem may be applied to expressing, by a series, any given root of a binomial, or any given root of any given power, or the reciprocal of any of these. In other words, the theorem holds good whether $n$ be an integer or a fraction, positive or negative. It is only, however, in the case of $n$ being a positive integer, that it is proposed to apply or demonstrate the theorem in this place: the demonstration, in the other cases, being of a less elementary character, may be reserved for a subsequent stage in the student's progress.

Dem. We find, by actual multiplication,

$$(x+a)(x+b) = x^2 + (a+b)x + ab, \text{ and}$$
$$(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc.$$

If we continue this process as far as we please, introducing successively the factors $x+d$, $x+e$, &c., we perceive that a certain order holds good throughout, both among the powers of $x$ and among the coefficients, viz. that the highest power of $x$ is in the first term, its exponent, in that term, being the same as the number of factors, and its exponent, in the other terms, decreasing by 1 in each successive term. As we also perceive, the coefficient of $x$, in the first term, is 1; in the second term it is the sum of the quantities $a$, $b$, $c$, &c. taken singly; in the third, it is the sum of all their products taken two by two; and so on till we come to the last term, which is the product of all the letters $a$, $b$, $c$, &c. without $x$.

That this *must* be the case, will be seen by observing the change produced by one multiplication, thus,—

$$\begin{array}{l} x^3+(a+b+c)x^2+(ab+ac+bc)x+abc \\ x+d \\ \hline x^4+(a+b+c)x^3+(ab+ac+bc)x^2+abcx \\ \qquad\qquad +dx^3+(ad+bd+cd)x^2+(abd+abc+bcd)x+abcd. \\ \hline x^4+(a+b+c+d)x^3+\left\{\begin{matrix} ab+ac+bc \\ +ad+bd+cd \end{matrix}\right\}x^2+\left\{\begin{matrix} abc+abd \\ +acd+bcd \end{matrix}\right\}x \\ +abcd. \end{array}$$

Here we perceive that the number of terms increases one at each mult$^{n}$, the last term being necessarily free from $x$, and the previous term, which was the last in the previous stage, containing the first power of $x$, every previous term having the exponent of $x$ raised a unit at each mult$^{n}$. Since, $\therefore$, the exponent of $x$ was 1 in the first term at the commencement, it must be $n$ when $n$ factors are multiplied together; and, since the exponents decreased by 1 in each successive term in the first and second products, they must continue to do so in every subsequent product. The product of $n$ factors must $\therefore$ be $x^n + Px^{n-1} + Qx^{n-2} + Rx^{n-3} +$ &c., $P$, $Q$, $R$, &c. being coefficients not yet determined.

If we return to the consideration of the process of multiplication, we shall see that, if the coefficient of the second term was $a + b + c$ in the product of three factors, or the sum of all the quantities, $a$, $b$, $c$, taken singly, so, since $d$ is added to it in the next mult$^{n}$, it becomes $a + b + c + d$, or still the sum of all the quantities, $a$, $b$, $c$, $d$, taken singly, and, since the same process is continued at every subsequent mult$^{n}$, the coef$^{ts}$ of the second term will always be the sum of the quantities, $a$, $b$, $c$, &c. taken singly.

Then, since, in the product of three factors, *the coef$^{t}$ of the third term* was the sum of all the comb$^{ns}$ of $a$, $b$, &c. taken 2 by 2, it must continue to be so, for, at each mult$^{n}$, we have added to it all the comb$^{ns}$ of the single letters with the new quantity.

Again, since, in the product of three factors, *the coef$^{t}$ of the fourth term* was the sum of all the comb$^{ns}$ of $a$, $b$, &c., taken 3 together, it must continue to be so, since, in each mult$^{n}$, we have added to it all the comb$^{ns}$ of every two of these letters with the new letter.

Since *the last term*, in the product of three factors, was the product of all the letters, $a$, $b$, $c$, the last term must continue to be the product of all the letters; for it is multiplied by a new letter at each successive stage.

In general, if, in any product, the coef$^t$ of the $r$ term is the sum of all the comb$^{ns}$ of all the letters, taken $r-1$ together, it will continue to be so in all subsequent products (See Ch. xxxiii. Th. 3, Cor. 4); and it was so in the product of 3 factors and is so with every new term introduced at the close, $\therefore$ it will be so in the product of 4 factors, and consequently of 5, and so on as far as we please.

If we now suppose all the quantities, $a$, $b$, $c$, &c. to be equal, the product of $n$ factors, $(x+a)(x+b)$ &c., becomes $(x+a)^n$. At the same time $P$, $=a+b+c+$ &c., becomes $na$. Also $ab$, $ac$, $bc$, &c., become severally $a^2$, their number being ${}^nC_2 = \frac{n(n-1)}{1.2}$. $\therefore Q \doteq \frac{n(n-1)}{1.2}a^2$. The comb$^{ns}$ of the letters in threes, $abc$, $abd$, &c., severally become $a^3$, their number being ${}^nC_3 = \frac{n(n-1)(n-2)}{1.2.3}$, and $R \doteq \frac{n(n-1)(n-2)}{1.2.3}a^3$. In like manner we find $S = \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}a^4$; and so on. Therefore $(x+a)^n$

$$\doteq x^n + nx^{n-1}a + \frac{n(n-1)}{1.2}x^{n-2}a^2 + \frac{n(n-1)(n-2)}{1.2.3}x^{n-3}a^3 + \text{\&c.}$$

If $a$ be negative, $a$, $a^3$, $a^5$, &c. will severally be negative, and the terms containing them negative; while $a^2$, $a^4$, &c., and the terms containing them, will all be positive. $\therefore$

$$(x-a)^n \doteq x^n - nx^{n-1}a + \frac{n(n-1)}{1.2}x^{n-2}a^2 - \frac{n(n-1)(n-2)}{1.2.3}x^{n-3}a^3$$

+ &c. The same thing may be seen by observing that in raising $x-a$ to any power, if the signs are alternately positive and negative at any stage, they *must* continue to be so; and, since they are so in the first and second powers, they must therefore be so in the third, fourth, and so on without limit.

If, then, we unite the expressions for $(x+a)^n$ and $(x-a)^n$ into one, we have the theorem as given at the commencement of the chapter.

Cor 1. It will be seen, from the theorem, that, when the exponent of $a$ is any number, say $m$, that of $x$ is $n-m$. The sum of the exponents of $x$ and $a$ will therefore always be the same, or the terms will all be *of the same dimensions*, as it is expressed; and, when the exponent of $a$ becomes $n$, that of $x$ becomes $n-n$, or 0, $x$ consequently disappearing.

COR. 2. In applying the theorem to the numerical computation of the coefficients, when $n$ is a given number, we are not under the necessity of computing each coefficient independently of the others, but may save much labour, if we observe that each may be deduced from that which precedes it, by multiplying the latter by the exponent of $x$ in the same term, and dividing by the number of that term,—that is, by 3 if it is the 3d term, and so on.

COR. 3. If we put $r$ for the number of any term, and $e$ for the exponent of $x$ in that term, it will be perceived, from Cor. 2, that, in passing from that to the next term, the coefficient will increase so long as $e$ is $> r$, but that, since $e$ constantly decreases and $r$ increases, $e$ must ultimately become less than $r$, and then the coefficients will continually become less until $e$ becomes $=0$, rendering every subsequent coefficient $=0$, and consequently terminating the series. The last term will therefore be that in which the exponent of $a$ is $n$, and that of $x$ is $n-n$, or 0.

Since $x \pm a$ consists of two terms and a new term is added at every multiplication, there must always be one term more than the number of factors, or the final value of $r$ is $n+1$.

COR. 4. Since, in every term, $e = n-(r-1) = n-r+1$, it follows that if, in any term, $r = \frac{1}{2}(n+1)$, or $2r = n+1$, $r = n-r+1$, and $\therefore e = r$. Now this will always occur in some one term when $n$ is an odd number, or $n+1$ even, since then $\frac{1}{2}(n+1)$ is an integer, and, as $r$ continually increases up to $n+1$, it must, at one stage, become $=\frac{1}{2}(n+1)$, and $e=r$. In this case the coefficient of the $r+1$ term will be the same as that of the $r$ term. But this can never occur if $n$ is even, or $n+1$ odd, because then $\frac{1}{2}(n+1)$, not being an integer, can never be $=r$.

Since the coefficient of the $r+1$ term is ${}^nC_r$, and $\therefore$ the coef^t^ of the $n-r+1$ term, ${}^nC_{n-r}$, and since ${}^nC_r = {}^nC_{n-r}$ (Ch. XXXIII. Th. 3, Cor. 3), it follows that the coef^t^ of the $r+1$ term is always equal to that of the $n-r+1$ term; and since the $n+1$ term is the last, we shall have the coef^t^ of the first term = that of the last, the coef^t^ of the second = that of the last but one, and so on; and, since the first term is $x^n$ with the exponent of $x$ diminishing by 1 in every succeeding term, and the last term is $a^n$, with the exponent of $a$ diminishing by 1 in each preceding term, the last part of the series must always be an exact counterpart of the first, term for term, $a$ taking the place of $x$, and $x$ of $a$, a single term occupying the middle place when $n$ is even and the number of terms odd. All this might be anticipated since $(x+a)^n$ must be $=(a+x)^n$.

EXAMPLE 1. What is the fifth power of $a-b$?

Here the terms, without the coef$^{ts}$, will be $a^5$, $a^4b$, $a^3b^2$, $a^2b^3$, $ab^4$, $b^5$; the coef$^{ts}$ will be 1, 5, $\frac{5\times 4}{2}=10$, $\frac{10\times 3}{3}=10$, $\frac{10\times 2}{4}=5$, $\frac{5\times 1}{5}=1$; and the signs of the terms will be alternately $+$ and $-$.

$$\therefore (a-b)^5 \doteq a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5.$$

EXERCISE 1. Raise $a+b$ to the third power, and $a-b$ to the sixth.

Ans. $a^3+3a^2b+3ab^2+b^3$, and $a^6-6a^5b+15a^4b^2-20a^3b^3+15\,a^2b^4-6ab^5+b^6$.

A. Express $(x-y)^4$ and $(x+y)^7$ each by a series.

B. Expand $(c-d)^{10}$ into a series.

C. What are the first four and the last four terms of the expanded value of $(y-z)^{100}$?

NOTE 1. When the terms of the given binomial consist partly of numbers and partly of letters, we must be careful, in the first stage of the process, to keep the coefficients of the given terms distinct from those which arise from the application of the theorem. Afterwards they may be united.

EXAMPLE 2. Raise $2c-3y$ to the fourth power.

$$\begin{aligned}(2c-3y)^4 &\doteq (2c)^4-4(2c)^3 3y+6(2c)^2(3y)^2-4(2c)(3y)^3+(3y)^4\\ &=16c^4-4.8c^3.3y+6.4c^2.9y^2-4.2c.27y^3+81y^4\\ &=16c^4-96c^3y+216c^2y^2-216cy^3+81y^4.\end{aligned}$$

EXERCISE 2. Expand $(3y-2z)^5$.

Ans. $243y^5-810y^4z+1080y^3z^2-720y^2z^3+240yz^4-32z^5$.

D. Give the expanded value of $(4a+5b)^6$.

NOTE 2. The same method must be followed if one of the terms of the given binomial is a number without a letter.

EXERCISE 3. Expand $(x-5)^5$.

Ans. $x^5-25x^4+250x^3-1250x^2+3125x-3125$.

E. Expand $(x+2)^{10}$ into a series.

4. The same with $(2+y)^4$.

Ans. $16+32y+24y^2+8y^3+y^4$.

F. Also with $(3-z)^5$.

5. The same with $(1+a)^6$.

Ans. $1+6a+15a^2+20a^3+15a^4+6a^5+a^6$.

G. Also with $(1-x)^{12}$.

H. And with $(1+2x)^7$ and $(1+\frac{1}{5}y)^5$.

6. Expand $(2a+\frac{1}{2}b)^8$ into a series.

Ans. $256a^8+512a^7b+448a^6b^2+224a^5b^3+70a^4b^4+14a^3b^5$
$+\frac{7}{4}a^2b^6+\frac{1}{8}ab^7+\frac{1}{256}b^8$.

K. Expand $(3x-\frac{1}{3}y)^9$ and $(\frac{1}{3}c-\frac{1}{4}d)^4$.

NOTE 3. If letters with exponents occur in the terms of the given binomial, the same precaution is necessary with them, viz. *at first* to keep the exponents of the given terms distinct from those which arise from the theorem.

EXAMPLE 3. Expand $(y^2+z^3)^5$.

$$(y^2)^5+5(y^2)^4z^3+10(y^2)^3(z^3)^2+10(y^2)^2(z^3)^3+5(y^2)(z^3)^4+(z^3)^5$$
$$=y^{10}+5y^8z^3+10y^6z^6+10y^4z^9+5y^2z^{12}+z^{15}.$$

EXERCISE 7. Expand $(b^5-c^4)^3$ and $(x^2+y^2)^6$.

Ans. $b^{15}-3b^{10}c^4+3b^5c^8-c^{12}$, and $x^{12}+6x^{10}y^2+15x^8y^4$
$+20x^6y^6+15x^4y^8+6x^2y^{10}+y^{12}$.

L. Expand $(d^3+e^5)^4$ and $(y^3-z^4)^7$.

8. The same with $(a^{\frac{1}{2}}+b^{\frac{1}{3}})^6$.

Ans. $a^3+6a^{\frac{5}{2}}b^{\frac{1}{3}}+15a^2b^{\frac{2}{3}}+20a^{\frac{3}{2}}b+15ab^{\frac{4}{3}}+6a^{\frac{1}{2}}b^{\frac{5}{3}}+b^2$.

M. The same with $(a^{\frac{1}{2}}-x^{\frac{1}{3}})^6$.

9. The same with $(4c^4+\frac{1}{4}d^{\frac{1}{4}})^4$.

Ans. $256c^{16}+256c^{12}d^{\frac{1}{4}}+6c^8d^{\frac{1}{2}}+\frac{1}{4}c^4d^{\frac{3}{4}}+\frac{1}{256}d$.

N. Also with $(\frac{1}{2}x^2-2z^{\frac{1}{2}})^8$.

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# CHAPTER XXXV.

## RATIO AND PROPORTION.

(*Continued from Chapter* XV.)

### THEOREM I.

*When four Quantities are proportional, if the first is equal to the second, the third is equal to the fourth; if greater, greater; and if less, less.*

DEM. If $a:b::c:d$, $\frac{a}{b}=\frac{c}{d}$ (Ch. XV. Def. 1). If, then, $a \doteq b$, $\frac{a}{b} \doteq 1$, and $\therefore \frac{c}{d} \doteq 1$. If $a \gtrdot b$, $\frac{a}{b} \gtrdot 1$, $\frac{c}{d} \gtrdot 1$, and $c > d$. If $a \lessdot b$, $\frac{a}{b} \lessdot 1$, $\frac{c}{d} \lessdot 1$, and $c < d$.

NOTE. The dot attached to the symbol > or <, is read *is*, as in the case of the symbol of equality.

COR. Since $a : b :: c : d$, alternately $a : c :: b : d$, and $\therefore$ if $a$ is $=$, $>$, or $< c$, $b$ is $=$, $>$, or $< d$.

## THEOREM II.

*Equal Quantities have the same Ratio to the same Quantity, and the same has the same Ratio to equal Quantities.*

That is, if $a$ and $b$ be two equal quantities, and $c$ any other quantity, then $a : c :: b : c$, and $c : a :: c : b$.

DEM. Since $a \doteq b$, $\frac{a}{c} \doteq \frac{b}{c}$, and $\therefore a : c :: b : c$; and, inversely $c : a :: c : b$.

COR. Hence equal quantities have the same ratio to equal quantities. That is, if $a \doteq c$ and $b = d$, then $a : b :: c : d$.

## THEOREM III.

*Quantities which have the same Ratio to the same Quantity are equal; and those to which the same has the same Ratio are equal.*

DEM. If $a : c :: b : c$, then $\frac{a}{c} \doteq \frac{b}{c}$, and $a = b$.

Again, if $c : a :: c : b$, inversely $a : c :: b : c$, and, by the preceding proof, $a \doteq b$.

DEF. 1. If $a$ contains $b$ oftener than $c$ contains $d$, or if $\frac{a}{b} \gtrdot \frac{c}{d}$, then $a$ is said to have a *Greater Ratio* to $b$ than $c$ has to $d$; or, as it is written, $a : b \gtrdot c : d$.

EXERCISE A. Whether is the ratio of 4 to 3 or that of 3 to 2 the greater?

B. Which is the greater ratio, 16 : 20 or 9 : 12?

C. Of the two ratios $a + b : a \backsim b$ and $a^2 + b^2 . a^2 \backsim b^2$, which is the greater?

Cor. 1. Hence a ratio is increased by increasing the antecedent or diminishing the consequent; and a ratio is diminished by diminishing the antecedent or increasing the consequent.

Cor. 2. Hence, also, the greater of two unequal quantities has a greater ratio to the same than the less has, because it contains it oftener; and for the same reason the same quantity has a greater ratio to the less than it has to the greater.

Exercise D. Whether is the ratio of 10 to 3 or of 11 to 3 the greater, and whether that of 2 to 5 or of 2 to 7?

Cor. 3. Hence also that quantity which has a greater ratio than another has to the same quantity is the greater of the two; and that quantity to which the same has a greater ratio than it has to another quantity is the less of the two. For if $a : c > b : c$, $\frac{a}{c} > \frac{b}{c}$ by the definition. $\therefore$ $a > b$. But if $c : b > c : a$, $\frac{c}{b} > \frac{c}{a}$, and $\therefore b < a$.

## Theorem IV.

*Ratios that are equal to the same Ratio are equal to one another.*

If $a : b :: c : d$, and $c : d :: e : f$, then $a : b :: e : f$.

Dem. Since $a : b :: c : d$, $\frac{a}{b} \doteq \frac{c}{d}$, and, since $c : d :: e : f$, $\frac{c}{d} \doteq \frac{e}{f}$. $\therefore \frac{a}{b} \doteq \frac{e}{f}$, and $a : b :: e : f$.

## Theorem V.

*If the first has to the second the same Ratio which the third has to the fourth, but the third to the fourth a greater Ratio than the fifth has to the sixth, the first shall also have to the second a greater ratio than the fifth has to the sixth.*

Thus, if $a : b \doteq c : d$, and $c : d > e : f$, then $a : b > e : f$.

Dem. Since $a : b \doteq c : d$, $\frac{a}{b} \doteq \frac{c}{d}$; since $c : d > e : f$, $\frac{c}{d} > \frac{e}{f}$ (Def. 1). $\therefore \frac{a}{b} > \frac{e}{f}$, and (Def. 1) $a : b > e : f$.

### Theorem VI.

*When four Quantities are Proportionals, the Sum or Difference of the first and second is to the second, as the Sum or Difference of the third and fourth is to the fourth.*

Dem. If $a : b :: c : d$, $\frac{a}{b} \doteq \frac{c}{d}$. But $\frac{b}{b} \doteq \frac{d}{d}$. Adding equals to equals, $\frac{a+b}{b} \doteq \frac{c+d}{d}$, or $a+b : b :: c+d : d$.

Taking equals from equals $\frac{a \backsim b}{b} \doteq \frac{c \backsim d}{d}$, or $a \backsim b : b :: c \backsim d : d$.

Cor. 1. Since, if $a : b :: c : d$, $-a : b :: -c : d$, and $a : -b :: c : -d$, we may at any time write $-$ for $\backsim$ in the first or second term provided we also do so in the third or fourth, and that without considering whether $a > b$, and $c > d$, or less. Our proposition may therefore be expressed, *generally*, thus

$$a \pm b : b :: c \pm d : d,$$

provided that we use corresponding signs, viz. + with + and − with −; a proviso which must also be observed in the following corollaries.

Cor. 2. Since the proportion $a : b :: c : d$ becomes inversely $b : a :: d : c$, $\therefore$, by the theorem,

$$a \pm b : a :: c \pm d : c,$$

or, inversely, $a : a \pm b :: c : c \pm d$,

and, alternately, $a : c :: a \pm b : c \pm d$.

Note. When we have given the proportion $a : b :: c : d$, and obtain from it the proportions

$$a+b : b :: c+d : d,$$
$$a-b : b :: c-d : d, \text{ and}$$
$$a : a-b :: c : c-d,$$

we are said to do so in the first case *by composition*, or *componendo*, in the second, *by division*, or *dividendo*, and in the third, *by conversion*, or *convertendo*. These terms, though much used, are anything but appropriate.

COR. 3. Since the given proportion, $a : b :: c : d$, becomes, alternately, $a : c :: b : d$, we obtain from this, by proceeding as we did with the given proportion,

$$a \pm c : c :: b \pm d : d,$$
$$a \pm c : a :: b \pm d : b,$$
$$a : a \pm c :: b : b \pm d, \text{ and}$$
$$a : b :: a \pm c : b \pm d, \text{ and,}$$

from Th. 4, $c : d :: a \pm c : b \pm d.$

EXERCISE A. Having given the proportion 15 : 10 :: 9 : 6, deduce from it all the proportions that can be obtained by this theorem and its corollaries.

### THEOREM VII.

*When four Quantities are proportionals, the Sum of the first and second is to their Difference as the Sum of the third and fourth to their Difference.*

DEM. If $a : b :: c : d$, then, by Th. 6, Cor. 2, $a+b : a :: c+d : c$, and $a : a \backsim b :: c : c \backsim d$.

From the former of these proportions, $\frac{a+b}{a} \doteq \frac{c+d}{c}$, and, from the latter, $\frac{a}{a \backsim b} \doteq \frac{c}{c \backsim d}$.

Multiplying equals by equals,

$$\frac{a+b}{a} \times \frac{a}{a \backsim b} = \frac{c+d}{c} \times \frac{c}{c \backsim d}, \text{ or } \frac{a+b}{a \backsim b} = \frac{c+d}{c \backsim d}.$$
$$\therefore a+b : a \backsim b :: c+d : c \backsim d.$$

COR. 1. Inversely, $a \backsim b : a+b :: c \backsim d : c+d$; and, alternately, $a+b : c+d :: a \backsim b : c \backsim d$.

EXERCISE A. Having given the proportioh 6 : 8 : : 9 : 12, deduce from it three other proportions, by this theorem and Cor. 1.

COR. 2. Taking $m$, $n$, $p$ and $q$ to represent any four numbers; then, if $a : b :: c : d$, we have (xv. 6) $ma : nb :: mc : nd$, and thence (Th. 6, Cor. 2) $ma \pm nb : mc \pm nd :: ma : mc :: a : c$; and, in like manner, $pa \pm qb : pc \pm qd :: a : c$. $\therefore$ (Th. 4)

$$ma \pm nb : mc \pm nd :: pa \pm qb : pc \pm qd.$$

This form is so general that it includes Theorems 6 and 7, with all their corollaries, besides innumerable other cases, as will appear by giving particular values (including 1 and 0) to $m$, $n$, $p$ and $q$. It is to be observed, however, that the signs, $+$ and $-$, must not be used indiscriminately, but that *they must be the same in the terms of the same ratio*, although the positive sign may be used to connect the terms of the one ratio, and the negative sign to connect those of the other.

EXERCISE B. Having given the proportion $12 : 8 :: 15 : 10$, and taking the numbers 4, 3, 2, 1, for $m$, $n$, $p$, $q$, deduce all the proportions that can be obtained by Cor. 2, with all the possible variations of sign under the limitation laid down.

DEF. 2. *Any number of quantities are said to be Proportionals*, when the first is to the second as the third to the fourth, and the first to the second, or the third to the fourth, as the fifth to the sixth, and so on.

## THEOREM VIII.

*If any Number of Quantities are Proportionals, as any one of the Antecedents is to its Consequent, so is the Sum of all the Antecedents to the Sum of all the Consequents.*

Thus, if $a : p :: b : q :: c : r$, then $a : p :: a+b+c : p+q+r$.

DEM. Since $a : p :: b : q$, $aq \doteq bp$ (XV. 1), and, since $a : p :: c : r$, $ar \doteq cp$; and $ap \doteq ap$. Adding equals to equals, $ap+aq+ar \doteq ap+bp+cp$, or $a \times (p+q+r) \doteq p \times (a+b+c)$. $\therefore$ (XV. 3) $a : p :: a+b+c : p+q+r$. And in the same manner we demonstrate the theorem whatever be the number of proportionals.

EXERCISE A. Since $2 : 3 :: 4 : 6 :: 6 : 9 :: 8 : 12$, obtain from this a single proportion.

COR. If $a : b :: c : d$
and $e : b :: f : d$, then
$a+e : b :: c+f : d$.

For, by alternation, $a : c :: b : d$, and $e : f :: b : d$. $\therefore$ (Th. 4) $a : c :: e : f$; and, by this theorem, $a : c :: a+e : c+f$. But $a : c :: b : d$. $\therefore a+e : c+f :: b : d$; or, alternately $a+e : b :: c+f : d$.

EXERCISE B. From the two proportions, $6 : 12 :: 10 : 20$, and $9 : 12 :: 15 : 20$, form one, by this corollary.

### THEOREM IX.

*If four Quantities are Proportionals, the Sum of the greatest and least is greater than the Sum of the other two.*

DEM. If $a : b :: c : d$, then, since $ad \doteq bc$, if $a$ is the greatest, $d$ is the least. But (Th. 6, Cor. 3) $a : b :: a - c : b - d$. $\therefore$ (Th. 1) if $a > b$, $a - c > b - d$. Adding $c + d$ to each of these, $a + d > b + c$.

In like manner it may be proved that if $b$ is the greatest and consequently $c$ the least, $b + c > a + d$.

### THEOREM X.

*If there be any Number of Quantities, and as many others, which, taken two and two in Order, have the same Ratios, the first shall have to the last of the first Quantities the same Ratio which the first of the others has to the last.*

DEM. *First*, let there be three quantities, $a$, $b$, $c$, and three others, $p$, $q$, $r$, such that $a : b :: p : q$, and $b : c :: q : r$. Then (XV. Def. 1) $\frac{a}{b} \doteq \frac{p}{q}$, and $\frac{b}{c} \doteq \frac{q}{r}$. $\therefore$, multiplying equals by equals,

$$\frac{a}{b} \times \frac{b}{c} \doteq \frac{p}{q} \times \frac{q}{r}, \text{ or } \frac{a}{c} \doteq \frac{p}{r}.$$

Hence $a : c :: p : r$.

*Next*, let there be four quantities, $a$, $b$, $c$, $d$, and four others, $p$, $q$, $r$, $s$, such that $a : b :: p : q$, $b : c :: q : r$, and $c : d :: r : s$. Then, taking the first three of each rank, viz. $a$, $b$, $c$, and $p$, $q$, $r$, we have, by the previous case, $a : c :: p : r$. But $c : d :: r : s$. $\therefore$, again by the first case, $a : d :: p : s$.

In this way we may advance successively to ranks of five, six, and any higher number of quantities.

NOTE. This theorem is usually referred to briefly by the expression, *ex æquo*, *ex æquali*, *ex æquali distantia*, or *by direct equality*.

## THEOREM XI.

*If there be any Number of Quantities, and as many others, which, taken two and two in a cross Order, have the same Ratios, the first shall have to the last of the first Quantities, the same Ratio which the first of the others has to the last.*

DEM. *First*, let there be three quantities, $a$, $b$, $c$, and three others, $p$, $q$, $r$, such that $a : b :: q : r$, and $b : c : p : q$. Then $\frac{a}{b} = \frac{q}{r}$, and $\frac{b}{c} = \frac{p}{q}$. $\therefore$, multiplying equals by equals $\frac{a}{b} \times \frac{b}{c} = \frac{q}{r} \times \frac{p}{q}$, or $\frac{a}{c} = \frac{p}{r}$, and consequently $a : c :: p : r$.

*Next*, let there be four quantities, $a$, $b$, $c$, $d$, and four others, $p$, $q$, $r$, $s$, such that $a : b :: r : s$, $b : c :: q : r$, and $c : d :: p : q$. Then, setting aside $d$ and $p$, we have three quantities $a$, $b$, $c$, and three others, $q$, $r$, $s$, such that $a : b :: r : s$, and $b : c :: q : r$. $\therefore$, by the first case, $a : c :: q : s$; and by hypothesis, $c : d :: p : q$. $\therefore$, again by the first case, $a : d :: p : s$.

In this way we may advance successively to ranks of five, six, and any higher number of quantities.

This theorem is often quoted by the terms, *ex æquali in proportione perturbata*, or *ex æquo perturbatè*, or *by indirect equality*.

DEF. 3. If the antecedents of any number of ratios be multiplied together, and the consequents together, the former product is said to have to the latter *the ratio compounded of all the separate ratios*. Thus the ratios $a : p$, $b : q$, $c : r$, when compounded together, make the ratio of $abc$ to $pqr$.

COR. Hence, of any number of quantities, the first has to the last the ratio compounded of the ratios of the first to the second, of the second to the third, and so on to the last. For (by the Def.) the ratio compounded of the ratios $a : b$, $b : c$, $c : d$, is $abc : bcd = a : d$.

EXERCISE 1. Compound the ratios $4 : 5$, $7 : 8$, and $15 : 14$, into one. Ans. $3 : 4$.

A. What is the ratio compounded of the ratios of 2 to 3, 5 to 6, and 9 to 10?

DEF. 4. Any number of *proportions* are said to be *com-*

*pounded together* when their corresponding ratios are compounded,—that is, when the corresponding terms are multiplied together.

### THEOREM XII.

*Ratios compounded of equal Ratios are equal.*

Thus, if $a : b :: c : d$ and $p : q :: r : s$, then $ap : bq :: cr : ds$.

DEM. Since $a : b :: c : d$ and $p : q :: r : s$, $\frac{a}{b} = \frac{c}{d}$, and $\frac{p}{q} = \frac{r}{s}$. $\therefore \frac{a}{b} \times \frac{p}{q} = \frac{c}{d} \times \frac{r}{s}$, or $\frac{ap}{bq} = \frac{cr}{ds}$, and $ap : bq :: cr : ds$.

In the same manner we proceed if more than two proportions are compounded together.

DEF. 5. Any number of quantities are said to be *Continued Proportionals* when the first is to the second as the second to the third, as the third to the fourth, and so on.

DEF. 6. When three quantities are continued proportionals, the first is said to have to the third the *Duplicate Ratio* of that which it has to the second; and the first is said to have to the second the *Subduplicate Ratio* of that which it has to the third.

EXERCISE A. To what number has 4 the duplicate ratio of that which it has to 6; and what is the subduplicate ratio of 20 to 45?

DEF. 7. When four quantities are in continued proportion, the first is said to have to the fourth the *Triplicate Ratio* of that which it has to the second; and the first is said to have to the second the *Subtriplicate Ratio* of that which it has to the fourth.

EXERCISE B. What is the triplicate ratio of 8 to 12, and what the subtriplicate ratio of 3 to 81?

### THEOREM XIII.

*When three Quantities are in continued Proportion, the first is to the third as the Square of the first to the Square of the second; and the first is to the second as the square Root of the first to the square Root of the third.*

DEM. If $a : b :: b : c$, since also $a : b :: a : b$, we have

(Th. 12) $a^2 : b^2 :: ab : bc :: a : c$. Then (xv. 7) $a : b :: \sqrt{a} : \sqrt{c}$.

Cor. Hence $a^2$ has to $b^2$ the duplicate ratio of $a : b$, and $\sqrt{a}$ has to the $\sqrt{c}$ the subduplicate ratio of $a : c$.

Exercise A. What is the duplicate ratio of 4 to 9, and what the subduplicate?

B. If $a : b :: b : c$, prove that $a^2 + b^2 : a :: b^2 + c^2 : c$.

### Theorem XIV.

*When four Quantities are continued Proportionals, the first is to the fourth as the Cube of the first to the Cube of the second; and the first is to the second as the Cube Root of the first to the Cube Root of the fourth.*

Dem. Since, evidently, $a : b :: a : b$,
and, by hypothesis, $a : b :: b : c$,
and also $a : b :: c : d$,
we have (Th. 12) $a^3 : b^3 :: abc : bcd :: a : d$ (xv. 6). Then (xv. 7) $a : b :: \sqrt[3]{a} : \sqrt[3]{d}$.

Cor. 1. Hence $a^3$ has to $b^3$ the triplicate ratio of $a$ to $b$, and $\sqrt[3]{a}$ has to $\sqrt[3]{d}$ the subtriplicate ratio of $a$ to $d$.

Cor. 2. In the same manner it may be proved that if $n$ quantities are continued proportionals, the first being $a$, the second $b$, and the last $l$, then $a : l :: a^n : b^n$, and $a : b :: \sqrt[n]{a} : \sqrt[n]{l}$.

Exercise A. What is the triplicate ratio of 8 to 27, and what the subtriplicate?

Def. 5. A ratio is said to be one *of Equality*, *of Greater Inequality*, or *of Less Inequality*, according as the antecedent is equal to, greater than, or less than, the consequent.

### Theorem XV.

*A Ratio of greater Inequality is diminished, and a Ratio of less Inequality is increased, by adding the same Quantity to both its Terms, provided that the terms of the given ratio and the quantity added are all positive.*

Thus, if 1 be added to each term of the ratio of 3 to 2, it becomes that of 4 to 3, which is less than the former since $\frac{4}{3} \lessdot \frac{3}{2}$ (Def. 1); and, if 1 be added to each term of the ratio of 2 to 3, it becomes that of 3 to 4, which is greater, since $\frac{3}{4} \gtrdot \frac{2}{3}$.

**Dem.** $\frac{a+b}{a} \doteq 1 + \frac{b}{a}$, and $\frac{a+b+c}{a+c} \doteq 1 + \frac{b}{a+c}$. But $1 + \frac{b}{a+c} \lessdot 1 + \frac{b}{a}$, when $a$, $b$, and $c$ are all positive quantities.

$\therefore \frac{a+b+c}{a+c} \lessdot \frac{a+b}{a}$, and $\therefore \frac{a+c}{a+b+c} \gtrdot \frac{a}{a+b}$.

Now if the ratio of greater inequality be expressed as $a + b : a$, and, if $c$ be added to each term, we have the ratio $a+b+c : a+c$, which is $< a+b : a$, since $\frac{a+b+c}{a+c} \lessdot \frac{a+b}{a}$, $a$, $b$, and $c$ being supposed positive.

Next, let the ratio of less inequality be that of $a : a+b$. Adding $c$ to each term, the ratio becomes $a+c : a+b+c$, which $\gtrdot a : a+b$, since $\frac{a+c}{a+b+c} \gtrdot \frac{a}{a+b}$.

**Cor.** Hence a ratio of greater inequality, whose terms are positive quantities, is increased, and one of less inequality is diminished by taking the same quantity from each term, so long as the quantity subtracted is not greater than either term.

**Exercise A.** Which is the greater ratio, $a + b : a - b$ or $a + 2b : a$, $a$ being $> b$?

B. Whether is the ratio $x + y : x + 2y$, or $x + 2y : x + 3y$ the greater, $x$ and $y$ being positive quantities?

**Def. 6.** If several quantities are multiplied by the same number, the products are said to be *Equimultiples* of the original quantities. Thus $na$, $nb$, $nc$, are equimultiples of $a$, $b$, and $c$.

## THEOREM XVI.

*If four Quantities are proportional, and if any Equimultiples be taken of the first and third, and also any Equimultiples of the second and fourth, then if the Multiple of the first is equal to that of the second, the Multiple of the third is equal to that of the fourth, if greater, greater, and if less, less.*

Thus, if $a : b :: c : d$, and if $ma$, $mc$ be any equimultiples of $a$ and $c$, and $nb$, $nd$, any equimultiples of $b$ and $d$, then, if $ma \doteq nb$, $mc \doteq nd$, if $ma \gtrdot nb$, $mc \gtrdot nd$, and if $ma \lessdot nb$, $mc \lessdot nd$.

DEM. Since $a : b :: c : d$, $ma : nb :: mc : nd$ (XV. 6). ∴ (Th. 1) if $ma \doteq nb$, $mc \doteq nd$, if $ma \gtrdot nb$, $mc \gtrdot nd$, and if $ma \lessdot nb$, $mc \lessdot nd$.

## THEOREM XVII.

*If four Quantities are so related to each other, that, on taking any Equimultiples whatever of the first and third, and any Equimultiples whatever of the second and fourth, the Multiple of the third is equal to that of the fourth, greater than it, or less, according as the Multiple of the first is equal to that of the second, greater than it, or less, then the first is to the second as the third to the fourth.*

Thus, if $a$, $b$, $c$, and $d$, are four quantities such that, on taking any equimultiples whatever of $a$ and $c$, as $ma$, $mc$, and any whatever of $b$ and $d$, as $nb$, $nd$, in all cases in which $ma \doteq nb$, $mc$ is also $= nd$, in all in which $ma \gtrdot nb$, $mc \gtrdot nd$, and in all in which $ma \lessdot nb$, $mc \lessdot nd$, then $a : b :: c : d$.

DEM. Let $c$, $d$, and $e$ be any three quantities; then, no matter how small $e$ may be, it is possible to take two numbers $m$ and $n$ such that $m(c+e)$ shall be $> nd$, but $mc$ not $> nd$. For, however small $e$ may be, $m$ may be taken so large that $me \gtrdot d$. Let it be so, and let $n$ be taken such that $nd$ is the smallest multiple of $d$ which is not $< mc$, and conseqʸ that $(n-1)d \lessdot mc$, or $nd - d < mc$. Then, adding $d$ to both, $nd \lessdot mc + d$, and still more is $nd < mc + me$, since $me \gtrdot d$. ∴ $nd \lessdot m(c+e)$, but not $< mc$, or $m(c+e) \gtrdot nd$, but $mc$ is not $> nd$.

To apply this to our purpose,—we shall have $a : b :: c : d$, if $\frac{a}{b} \doteq \frac{c}{d}$; but, if $\frac{a}{b}$ is not $= \frac{c}{d}$, one of them must be the greater. First suppose $\frac{a}{b}$ to be the greater, and let it be $= \frac{c+e}{d}$. Then $a : b :: c + e : d$, and ∴ (Th. 16) if $m(c+e) \gtrdot nd$, $ma \gtrdot nb$; but, by hypothesis, if $ma \gtrdot nb$, $mc \gtrdot nd$. ∴, whenever $m(c+e) \gtrdot nd$, $mc$ is also $\gtrdot nd$. But that is not true, for we have shown that two numbers $m$ and $n$ may be found such that $m(c + e) \gtrdot nd$ but $mc$ not $\gtrdot nd$.

Therefore $\frac{a}{b}$ is not $> \frac{c}{d}$; and in the same way it may be proved that $\frac{c}{d}$ is not $> \frac{a}{b}$. $\therefore \frac{a}{b} \doteq \frac{c}{d}$, and $a : b :: c : d$.

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# CHAPTER XXXVI.

## PROPORTIONAL VARIATION.

DEF. 1. Quantities which change their value are said to be *variable:* those which never change are called *constant.*

DEF. 2. If two variable quantities are so connected that when one of the two is changed in value the other is changed in the same proportion, *the one* is said *to vary directly as the other*, or simply *to vary as the other.*

Thus if $a$ and $a'$ are *any* two values of a variable quantity $A$, and $b$ and $b'$ the two corresponding values of another, $B$, and if $a : a' :: b : b'$, then $A$ is said to vary as $B$.

Thus also, in a garrison of a variable number of men, the daily consumption of provisions varies as the number of men, and the expense will usually vary as the quantity. If a circular plate of metal is uniformly expanded by heat, we say its circumference varies as its diameter.

That $A$ varies directly as $B$, is expressed symbolically thus; $A \propto B$. In this, although only two terms are expressed, we have really a proportion of four terms implied, for we mean that $a : a' :: b : b'$.

COR. 1. If the quantities $A$ and $B$ are of the same kind, we may state the proportion alternately, $a : b :: a' : b'$; and we can *always* do so *in Algebra*, since, in it, all quantities are treated as numbers. When $A \propto B$, then, the ratio of $A$ to $B$ is constant under every variation of the quantities themselves. And, conversely, when the ratio of any two quantities, $A$ and $B$, is constant, $A \propto B$.

COR. 2. If, $\therefore$, $A \propto B$, the quotient of $A$ by $B$ may be represented by a cons$^t$ number, $n$, or $A \div B \doteq n$, and $\therefore$ $A \doteq nB$. Conversely if $A \div B \doteq n$, or $A = nB$, $n$ being any cons$^t$ number, then $A \propto B$.

EXERCISE A. Prove that $3x-6a \propto x-2a$, and that $\frac{a^2-x^2}{a+x} \propto 3a-3x$.

B. Prove that the area of a triangle whose altitude is cons$^t$, varies as its base; but if the base is cons$^t$ and the altitude not so, the area of the triangle varies as its altitude.

C. If $3A+5B \doteq 7A-2B$, prove that $A \propto B$.

COR. 3. If $A \propto B$, and $B \propto C$, $A \propto C$. For (Cor. 2) $A \doteq mB$, and $B=nC$, $m$ and $n$ being cons$^t$ numbers. $\therefore$ $A \doteq mnC$, $mn$ being cons$^t$. $\therefore$ (Cor. 2) $A \propto C$.

COR. 4. Hence if $A \propto B$, $B \propto C$, $C \propto D$, and so on to any extent, any one of these quantities varies as any other.

COR. 5. If $A \propto B$, $nA \propto B$, $A \propto nB$, and $mA \propto nB$, $m$ and $n$ being any cons$^t$ numbers, integral or fractional. For, since $A \propto B$, $\frac{A}{B} \doteq q$, a cons$^t$ (Cor. 2), $\frac{nA}{B} \doteq nq$, a cons$^t$ $\frac{A}{nB} \doteq \frac{q}{n}$, a cons$^t$, and $\frac{mA}{nB} = \frac{mq}{n}$, which is also a cons$^t$.

COR. 6. If $A \propto C$, and $B \propto C$, $A \pm B \propto C$. For, let $A$ be $=mC$, and $B=nC$, then $A \pm B \doteq (m \pm n)C$.

EXERCISE D. Prove that if $A \propto B$, $A+B \propto A-B$.

E. Prove that if $A \propto B.C$, and $C \propto D$, $A \propto B.D$.

COR. 7. If $A \propto P$, and $B \propto Q$, $A.B \propto P.Q$. For, let $A$ be $=mP$, and $B=nQ$. Then $A.B=mn \times P.Q$.

COR. 8. Hence, if $A \propto P$, if $B \propto Q$, if $C \propto R$, and so on, $A.B.C\ldots \propto P.Q.R\ldots.$

COR. 9. Hence, making $A$, $B$, $C$, &c. equal, and $P$, $Q$, $R$, &c. equal, if $A \propto P$, $A^n \propto P^n$.

COR. 10. If $A \propto B$, and $A \propto C$, $A^2 \propto B.C$, and $A \propto \sqrt{(B.C)}$. For let $A$ be $=mB$ and $A=nC$, $A^2 \doteq mn \times B.C$, and $A=\sqrt{(mn)} \times \sqrt{(B.C)}$.

EXERCISE F. Prove that, in a circle, diam$^r$ $\propto \sqrt{(\text{rad}^s \times \text{circumf}^e)}$.

G. Prove that if $A \propto B$, $\frac{1}{A} \propto \frac{1}{B}$, and that $\frac{A}{B} \propto \frac{P}{Q}$ if $A \propto P$ and $B \propto Q$.

H. Prove that, if $A \propto B$, if $B \propto C$, and if $C \propto D$, $A \pm D \propto \sqrt{(B.C)}$.

K. If $A \propto B$, prove that $A^2 \pm B^2 \propto A.B$.

COR. 11. In like manner as in Cor. 10, it may be proved that, if $A$ varies as each of $n$ quantities, $B$, $C$, $D$, &c., $A^n \propto B.C.D...$, and $A \propto \sqrt[n]{(B.C.D...)}$.

Cor. 12. If $A^n \propto B^n$, $A \propto B$. For, let $A^n$ be $= mB^n$: then $A \doteq \sqrt[n]{m} \times B$.

Cor. 13. Hence, if $A \propto B$, $\sqrt[n]{A} \propto \sqrt[n]{B}$.

Cor. 14. If $A \propto B$, and $C$ is any other quantity, cons$^t$ or variable, $A.C \propto B.C$. This has been demonstrated in Cor. 5, when $C$ is cons$^t$. If $C$ is variable $A \propto B$, $C \propto C$, and ∴ (Cor. 7) $A.C \propto B.C$.

Cor. 15. If $A \propto B.C$, and if $B$ and $C$ are each capable of varying while the other remains cons$^t$, then $A \propto B$ when $C$ is cons$^t$, and $A \propto C$ when $B$ is cons$^t$. This is evident from Cor. 2; but the converse is not so evident, and will be demonstrated in Theorem 1.

Cor. 16. If $A \propto B.C$, then $B \propto \frac{A}{C}$ and $C \propto \frac{A}{B}$. For, since $A \propto B.C$, $A \doteq n.B.C$, $B \doteq \frac{1}{n} \times \frac{A}{C}$, and $C = \frac{1}{n} \times \frac{A}{B}$.

Def. 3. $A$ is said to vary as $B$ and $C$ *jointly* when it varies as their product, or when $A \propto B.C$.

Def. 4. $A$ is said *to vary inversely* as $B$, when $A$ varies directly as $\frac{1}{B}$, this relation being ∴ usually expressed symbolically thus, $A \propto \frac{1}{B}$. It may be expressed more simply thus, $A \infty B$.

Cor. 1. Since, when $A \infty B$, or $A \propto \frac{1}{B}$, $A \doteq n \times \frac{1}{B}$ (Def. 2, Cor. 2), ∴ $A \times B \doteq n$, and, conversely, if $A \times B \doteq n$, or $A \doteq n \times \frac{1}{B}$, $A \propto \frac{1}{B}$, or $A \infty B$. If, ∴, the product of any two variable quantities is cons$^t$, the one varies inversely as the other.

Exercise L. Prove that, if the area of a triangle is cons$^t$, the altitude $\infty$ the base, and if the solid content of a cone is cons$^t$, the altitude $\infty$ the square of the base.

M. Prove that if $A \infty B$, $B \infty A$.

Cor. 2. If $A \propto B$, and $B \infty C$, $A \infty C$. For $A \doteq mB$, and $B = n \times \frac{1}{C}$. ∴ $A \doteq mn \times \frac{1}{C}$, and (Cor. 1) $A \infty C$.

Cor. 3. If $A \infty B$, and $B \infty C$, $A \propto C$. For $A \doteq m \times \frac{1}{B}$, and $B = n \times \frac{1}{C}$, or $\frac{1}{B} = \frac{1}{n}C$. $\therefore A \doteq \frac{m}{n}C$, or $A \propto C$.

Cor. 4. If $A \infty B$, $nA \infty B$, $A \infty nB$, and $mA \infty nB$, $m$ and $n$ being any two cons$^{t}$ numbers, integral or fractional.

Cor. 5. If $A \infty C$, and $B \infty C$, $A \pm B \infty C$.

Cor. 6. If $A \infty P$, $B \infty Q$, $C \infty R$, and so on, $A.B.C\ldots \infty P.Q.R\ldots$.

Cor. 7. Hence if $A \infty P$, $A^n \infty P^n$.

Cor. 8. If $A \infty B$, and $A \infty C$, $A^2 \infty B.C$, and $A \infty \sqrt{(B.C)}$; and, generally, if $A$ varies inversely as each of $n$ quantities, $B$, $C$, $D$, &c., $A^n \infty B.C.D\ldots$, and $A \infty \sqrt[n]{(B.C.D\ldots)}$.

Cor. 9. If $A^n \infty B^n$, $A \infty B$; or, if $A \infty B$, $\sqrt[n]{A} \infty \sqrt[n]{B}$.

The preceding corollaries are proved in the same manner as the corresponding corollaries to Def. 2.

Exercise N. Prove that if $A \infty B.C$, $B \infty A.C$, and $C \infty A.B$.

O. Prove that if $A \propto B$ and $B \propto C$, $mA \pm nB \propto pC$.

Cor. 10. If $A \propto \frac{B}{C}$, and if $B$ and $C$ are each capable of varying while the other remains cons$^{t}$, then $A \propto B$ when $C$ is cons$^{t}$, and $A \infty C$ when $B$ is cons. For let $A$ be $= m\frac{B}{C}$, and, when $C$ is cons$^{t}$, $\frac{1}{C}$ may be expressed as $n$, and $A \doteq mnB$, or $A \propto B$. When $B$ is cons$^{t}$, $B$ may be expressed as $n$, and $A \doteq mn\frac{1}{C}$, or $A \infty C$.

This is what is meant when it is said by some writers, that, if $A \propto \frac{B}{C}$, $A \propto B$ and $A \infty C$. Such a statement however must be received with caution and only on the conditions laid down in the corollary. See Note 2 to Th. 1.

### THEOREM I.

*If, of three Quantities,* A, B, *and* C, A $\propto$ B *when* C *is constant, and* A $\propto$ C *when* B *is constant, then* A $\propto$ B.C *when all three are variable.*

Dem. When $B$ varies, let it change from $B$ to $mB$.

Then (since $A \propto B$) while $B$ becomes $mB$, $A$ becomes $mA$. Again, when $C$ varies, let $C$ become $nC$, then, since $A \propto C$, and conseq$^y$ $mA \propto C$, while $C$ becomes $nC$, $mA$ becomes $mnA$. In consequence, then, of both changes, $A$ becomes $mnA$.

But if $B$ becomes $mB$ and $C$ becomes $nC$, $B.C$ becomes $mn.B.C$.

While $A$, then, becomes $mnA$, $B.C$ becomes $mn.B.C$; and $A : mnA :: B.C : mn.B.C$. $\therefore A \propto B.C$ (Def. 2).

NOTE 1. On this theorem depends the principle of Compound Proportion in Arithmetic. If, for instance, the quantity of work done in a given time is proportional to the number of men employed, or varies as the number of men when the time is cons$^t$, and also varies as the time when the number of men is cons$^t$, it will vary as the product of the time and the number of men when neither is cons$^t$.

NOTE 2. We must carefully distinguish between the conditions of this theorem and those of Cor. 10 to Def. 2; otherwise the conclusions may appear contradictory. We must observe that, in the former, either $B$ or $C$ may vary while the other remains cons$^t$, whereas, in the latter, any change in $B$ necessarily involves a proportional change in $C$, since, if $A \propto B$, and $A \propto C$, $B \propto C$; so that one of the two cannot remain cons$^t$ while the other varies.

We have an example of the former case in an ellipse, whose area varies as the major axis when the minor is cons$^t$, and as the minor axis when the major is cons$^t$, and, conseq$^y$, by this theorem, varies as the product of the two when both change. Of the latter we have an instance in the circle, in which the diam$^r$ varies as the radius, and also as the circumf$^{ce}$; but, as the radius cannot vary without the circumf$^{ce}$ varying, the result is not that the diam$^r$ itself, but its square, varies as the product of the radius and circumf$^{ce}$, by Cor. 10 to Def. 2.

## THEOREM II.

*If, of three Quantities,* A, B, *and* C, $A \propto B$ *when* C *is constant, and* $A \mathrel{\reflectbox{$\propto$}} C$ *when* B *is constant,* $A \propto (B \div C)$ *when all three are variable.*

**DEM.** Since $A \propto B$, when $C$ is cons$^t$, and $A \propto \frac{1}{C}$ when

$B$ is cons$^{t}$, $\therefore$ (by Th. 1) $A \propto \left(B \times \frac{1}{C}\right)$, or $A \propto \frac{B}{C}$, when all three are variable.

**Example 1.** If $x^2 \infty y^3$, and two contemporaneous values of $x$ and $y$ are 1 and 2, what is the value of $x$ when $y$ is 4?

$x^2 \doteq n \times \frac{1}{y^3}$. $\therefore$, in the particular case mentioned, $1 \doteq n \times \frac{1}{8}$, and $n = 8$. But $n$ is the same in all cases: $\therefore$, when $y$ is 4, $x^2 = ny^3 \doteq 8 \times 64 = 2 \times 256$, and $x \doteq 16\sqrt{2}$.

**Exercise 1.** If $x^2 \propto y^3$, and two contemp$^{s}$ values of $x$ and $y$ are 8 and 2, what is the value of $y$ when $x$ is 4?

Ans. $\sqrt[3]{2}$.

**P.** If two contemp$^{s}$ values of $x$ and $y$ are 2 and 3, what is the value of $y$ when $x \doteq 10$; 1$^{st}$, when $x \propto y$; 2$^{dly}$, when $x \infty y$; 3$^{dly}$, when $x \propto y^2$; and, 4$^{thly}$, when $x \infty y^2$?

**Q.** If $A^2 + B^2 \propto A.B$, prove that $A + B \propto A - B$.

# CHAPTER XXXVII.

## HARMONICAL PROGRESSIONS.

**Def. 1.** *A Harmonical Progression* is that whose successive terms are the reciprocals of the corresponding terms of an arithmetical progression.

Thus, if 3, 5, 7, 9, or $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{6}$, 1, are the terms of an arith$^{l}$ prog$^{n}$, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, or 3, 2, $1\frac{1}{2}$, $1\frac{1}{5}$, 1, are the terms of a harm$^{l}$ prog$^{n}$.

**Note 1.** Such progressions received their name from the circumstance that, if the lengths of musical strings, of the same weight and tension, are to each other in the proportion of the numbers 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c., the shorter strings all vibrate harmoniously with the longest, the sounds of the former being called the harmonics of that of the latter.

**Def. 2.** Any three consecutive terms of a harmonical progression are said to be *in Harmonic Proportion.*

**Note 2.** Most questions relating to harm$^{l}$ prog$^{ns}$ may be resolved by the rules for arith$^{l}$ prog$^{ns}$, treating the terms of the one as the reciprocals of those of the other. We cannot, however, in that way or in any other, obtain a general rule for the sum of a harm$^{l}$ prog$^{n}$.

**Example 1.** The first and second terms of a harm prog$^n$ are 3, 2. What is the tenth term?

The recip$^{ls}$ of 3 and 2 are $\frac{1}{3}$ and $\frac{1}{2}$. The difference of these is $\frac{1}{6}$, and ∴ the tenth term of the arith$^l$ prog$^n$, $\frac{1}{3}$, $\frac{1}{2}$, &c., is $\frac{1}{3}+9\times\frac{1}{6}=\frac{11}{6}$. The recip$^l$ of this is $\frac{6}{11}$, which is the tenth term of the harm$^l$ prog$^n$.

**Exercise A.** Fill up the blanks in the following harm$^l$ prog$^{ns}$.

$$—, —, \tfrac{1}{3}, \tfrac{1}{4}, —, —, —;$$
$$\tfrac{2}{3}, \tfrac{1}{3}, —, —, —, —, —;$$
$$—, —, —, \tfrac{3}{7}, \tfrac{1}{3}, —, —;$$
$$—, —, \frac{a}{a-b}, \frac{a}{a+b}, —, —;$$
$$—, —, \frac{x}{x+3}, 1, —, —.$$

1. What is the fiftieth term of the harm$^l$ prog$^n$, 3, 1, &c.? Ans. $\frac{1}{33}$.

B. What is the 22d term of the prog$^n$, 5, 2, $1\frac{1}{4}$, &c.?

**Def. 3.** Two non-contiguous terms of a harmonical progression being given, the intermediate terms are called *Harmonic Means* between these.

**Example 2.** Insert five har$^e$ means between 3 and 1.

The recip$^{ls}$ of 3 and 1 are $\frac{1}{3}$ and 1, which are ∴ to be regarded as the first and seventh terms of an arith$^l$ prog$^n$. Then (xv. 2) $d=(1-\frac{1}{3})\div 6=\frac{2}{3}\times\frac{1}{6}=\frac{1}{9}$.

The arith$^l$ prog$^n$ will ∴ stand thus, $\frac{3}{9}$, $\frac{4}{9}$, $\frac{5}{9}$, $\frac{6}{9}$, $\frac{7}{9}$, $\frac{8}{9}$, $\frac{9}{9}$, and the harm$^l$ prog$^n$, 3, $\frac{9}{4}$, $\frac{9}{5}$, $\frac{9}{6}$, $\frac{9}{7}$, $\frac{9}{8}$, 1.

**Exercise C.** Insert one harm$^e$ mean between 42 and 21.

2. Insert three harm$^e$ means between 2 and 1. Ans. $1\frac{3}{5}$, $1\frac{1}{3}$, $1\frac{1}{7}$.

D. Insert two harm$^e$ means between 1 and $\frac{1}{3}$.

3. Compute two harm$^e$ means between $\frac{2a}{x-y}$, and $\frac{2a}{x+y}$.

Ans. $\frac{6a}{3x-y}$ and $\frac{6a}{3x+y}$.

E. Find one harm$^e$ mean between 1 and $\frac{x-y}{x+y}$.

**Example 3.** Of a harm$^l$ prog$^n$ the product of the first and fourth terms is $\frac{9}{64}$, and that of the second and third is $\frac{3}{32}$. What are the terms?

Let $\frac{1}{x-3y}$, $\frac{1}{x-y}$, $\frac{1}{x+y}$, and $\frac{1}{x+3y}$ represent the terms. Then $\frac{1}{x^2-9y^2} \doteq \frac{9}{64}$, and $\frac{1}{x^2-y^2} = \frac{3}{32}$; from which we find $y = \frac{2}{3}$ and $x = \frac{10}{3}$. Conseq$^y$ the prog$^n$ is $1 \div \frac{4}{3}$, $1 \div \frac{8}{3}$, $1 \div \frac{12}{3}$, and $1 \div \frac{16}{3}$, or $\frac{3}{4}$, $\frac{3}{8}$, $\frac{1}{4}$, and $\frac{3}{16}$. Or, if we take the negative values of $x$ and $y$, the progression will be $-\frac{3}{4}$, $-\frac{3}{8}$, $-\frac{1}{4}$, $-\frac{1}{16}$.

EXERCISE F. The second term of a harm$^l$ prog$^n$ is $\frac{1}{3}$ and the sum of the first and third is $\frac{3}{4}$. What are the three terms?

4. Of four numbers in harm$^l$ prog$^n$, the quotient of the first divided by the fourth is 4, and the sum of the second and third is $\frac{5}{4}$. What are the numbers?

Ans. $\frac{3}{2}$, $\frac{3}{4}$, $\frac{1}{2}$, $\frac{3}{8}$.

### THEOREM.

*Of any three consecutive Terms in a Harmonical Progression, the first is to the third as the Difference of the first and second to the Difference of the second and third.*

DEM. Let $a$, $b$, $c$ represent the three terms. Then their reciprocals, $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, being in arith$^l$ prog$^n$ $\frac{1}{a} \sim \frac{1}{b} \doteq \frac{1}{b} \sim \frac{1}{c}$, or $\frac{b \sim a}{ab} \doteq \frac{c \sim b}{bc}$; or $\frac{b \sim a}{c \sim b} = \frac{ab}{bc} = \frac{a}{c}$. $\therefore a : c :: b \sim a : c \sim b$.

EXERCISE. Prove that the difference of any two contiguous terms in a harm$^l$ prog$^n$ is to the difference of any other two contiguous terms, as the product of the former two to the product of the latter two.

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## CHAPTER XXXVIII.

## SURDS.

(*Continued from Chapters* XVI. *and* XXI.)

DEF. *A Binomial Surd* is a quantity consisting of two terms, one or both of which are surds. A *Trinomial Surd* consists of three terms, one or more of which are surds. These are both included under the more general name of *Compound Surds.*

### PROBLEM I.

*To express a rational Quantity in the Form of a Surd, or of a Power with any given fractional Exponent.*

RULE. Raise the given quantity to a power whose exponent is the reciprocal of the given fractional exponent, and over that power place the given fractional exponent.

DEM. $a^m \doteq (\sqrt[n]{a^m})^n$ (XVI. Def. Note) $= \left(a^{\frac{m}{n}}\right)^n$ (XXXI. Def. 1).

$\therefore a = \sqrt[m]{a^m} \doteq \sqrt[m]{\left\{\left(a^{\frac{m}{n}}\right)^n\right\}} = \left(a^{\frac{m}{n}}\right)^{\frac{n}{m}}$ (XXXI. Def. 1).

EXERCISE 1. Express $a$, 7, and $5x$ as square roots, or as powers with the exponent $\frac{1}{2}$.

Ans. $(a^2)^{\frac{1}{2}}$, $49^{\frac{1}{2}}$, $(25x^2)^{\frac{1}{2}}$, or $\sqrt{a^2}$, $\sqrt{49}$, $\sqrt{(25x^2)}$.

2. Express $b$, 3, and $4ay$ as cube roots, or as powers with the exp^t $\frac{1}{3}$.

Ans. $(b^3)^{\frac{1}{3}}$, $27^{\frac{1}{3}}$, $(64a^3y^3)^{\frac{1}{3}}$, or $\sqrt[3]{b^3}$, $\sqrt[3]{27}$, and $\sqrt[3]{(64a^3y^3)}$.

A. Express $cd$, 12, and $9z$ as powers with the exp^t $\frac{1}{2}$.

B. Put $ab^2$, 6, and $5c$, each into the form of a power with the exp^t $\frac{1}{3}$.

3. Reduce $x$, 9, and $4y^2$ into the form of powers, each with the fractional exp^t $\frac{2}{3}$.

Ans. $(x^{\frac{3}{2}})^{\frac{2}{3}}$, $27^{\frac{2}{3}}$, and $(8y^3)^{\frac{2}{3}}$.

C. Express $a$, 27, $b^2$, $c^3$, and $64d^3$ as powers having each the exp^t $\frac{3}{4}$.

4. Write $a^3+3a^2b+3ab^2+b^3$ as a power with its index $\frac{3}{2}$.

Ans. $(a^2+2ab+b^2)^{\frac{3}{2}}$.

D. Write $a^2-2ab+b^2$ as a power with its index $\frac{2}{3}$.

COR. 1. A rational coefficient of a surd may be introduced under the same exponent with the surd, by first changing the coefficient into a surd of the same exponent, by this problem, and then multiplying the two surds together, by Pr. 2 of Ch. XVI.

EXERCISE 5. Change $15\sqrt{x}$, $5a\sqrt[3]{y}$, $3\sqrt{2}$, $\frac{1}{3}\sqrt[3]{(3c)}$, $5a\sqrt[3]{(\frac{1}{10}ay)}$, and $4z^{\frac{2}{3}}$, into the form of entire surds.

Ans. $\sqrt{(225x)}$, $\sqrt[3]{(125a^3y)}$, $\sqrt{18}$, $\sqrt[3]{(\frac{1}{9}c)}$, $\sqrt[3]{(\frac{25}{2}a^4y)}$, and $(8z)^{\frac{2}{3}}$.

E. Express $7\sqrt{a}$, $\frac{1}{2}\sqrt{(2b)}$, $5\sqrt{(\frac{1}{5}c)}$, $3a^2\sqrt[3]{(2a^2d^2)}$, $\frac{1}{2}\sqrt[3]{(4x^2y)}$, and $27z^{\frac{3}{4}}$, as entire surds.

6. The same with $\frac{a-b}{a+b}\sqrt{\frac{a+b}{a-b}}$. Ans. $\sqrt{\frac{a-b}{a+b}}$.

F. Also $\frac{a}{b}\sqrt[3]{\frac{b}{a}}$, and $\frac{2x+3y}{2x-3y}\times\left(\frac{2x-3y}{2x+3y}\right)^{\frac{2}{3}}$.

Cor. 2. The rule for this problem is, of course, equally applicable when the given exponent is an integer.

Exercise 7. Express $2x$, $27y^2$, and $\frac{1}{9}z^3$ as powers having their respective indices 4, 3, and 2.

Ans. $(2^{\frac{1}{4}}x^{\frac{1}{4}})^4$, $(3y^{\frac{2}{3}})^3$, and $(\frac{1}{3}z^{\frac{3}{2}})^2$.

G. Express $a$, $8b^2$, and $\frac{1}{4}c^3$ as powers having their respective exp$^{ts}$ 4, 3, and 2.

Cor. 3. By the same rule a quantity having a given *fractional* exponent is changed into a quantity having another given exponent, integral or fractional.

Example. Change $32a^{\frac{2}{3}}$ into a quantity having the index $\frac{5}{6}$.

$$32a^{\frac{2}{3}} \doteq \{(32a^{\frac{2}{3}})^{\frac{6}{5}}\}^{\frac{5}{6}} = \{(32a)^{\frac{4}{5}}\}^{\frac{5}{6}} = (16a^{\frac{4}{5}})^{\frac{5}{6}}.$$

Exercise 8. Change $a^{\frac{2}{3}}$ and $3b^{\frac{1}{2}}$ into quantities having their exp$^{ts}$ each $\frac{1}{6}$. Ans. $(a^4)^{\frac{1}{6}}$ and $(27b^3)^{\frac{1}{6}}$.

H. Change $x^{\frac{2}{3}}$ and $(2y)^{\frac{3}{4}}$ into powers indicated by the exp$^t$ $\frac{1}{12}$.

K. Express $8a^{\frac{3}{4}}$ as a power with its exp$^t$ $\frac{3}{8}$.

Cor. 4. By the same rule any compound surd may be reduced to the form of an entire surd of any given order. Thus, if it were required to reduce $\sqrt{a}+\sqrt{b}$ into the form of an entire quadratic surd, we have merely to square it and then attach the sign of the square root. Thus, $\sqrt{a}+\sqrt{b} \doteq \sqrt{(\sqrt{a}+\sqrt{b})^2} = \sqrt{(a+2\sqrt{ab}+b)}$.

Exercise 9. Reduce $x-\sqrt{y}$, and $\sqrt{(2a)}+\sqrt{(3b)}$ to the form of entire quadratic surds.

Ans. $\sqrt{(x^2-2x\sqrt{y}+y)}$ and $\sqrt{(2a+2\sqrt{6ab}+3b)}$.

10. The same with $\sqrt{10}-\sqrt{5}$ and $\sqrt{\frac{4}{3}}-\sqrt{\frac{3}{4}}$.

Ans. $15-10\sqrt{2}$, and $\sqrt{\frac{1}{12}}$ or $\sqrt{4\frac{1}{12}}$.

L. The same with $\sqrt{c}+d$, $\sqrt{\frac{2}{3}x}-\sqrt{\frac{3}{2}y}$ and $2+\sqrt{3}$.

11. Reduce $\sqrt[3]{a}-\sqrt[3]{b}$ to the form of an entire cubic surd.

Ans. $\sqrt[3]{\{a-3\sqrt[3]{(a^2b)}+3\sqrt[3]{(ab^2)}-b\}}$.

12. The same with $1+\sqrt[3]{3}$.

Ans. $\sqrt[3]{(4+3\sqrt[3]{3}+3\sqrt[3]{9}}$.

13. The same with $3-\sqrt{5}$ and $\sqrt{2}+\sqrt{3}$.

Ans. $\sqrt[3]{(72-32\sqrt{5})}$ and $\sqrt[3]{(11\sqrt{2}+9\sqrt{3})}$.

M. The same with $\sqrt[3]{(2a)}+\sqrt[3]{(\frac{1}{2}b)}$, $3-\sqrt[3]{2}$, $4+\sqrt{3}$, $4-\sqrt{3}$, and $\sqrt{5}-\sqrt{3}$.

## PROBLEM II.

*To reduce Surds of different Kinds to Surds of the same Kind.*

**RULE.** Reduce their fractional exponents to a common denominator.

**DEM.** In Ch. XVI. surds have been defined to be of the same kind when the root is the same in both. It follows, therefore, that surds are of the same kind when, on being expressed as powers, their fractional exp<sup>ts</sup> have the same den<sup>r</sup>; for the den<sup>r</sup> indicates the root.

**EXAMPLE 1.** Reduce $a^{\frac{1}{2}}$ and $a^{\frac{2}{3}}$ to surds of the same kind.
The com<sup>n</sup> den<sup>r</sup> is 6. Then $a^{\frac{1}{2}} \doteq a^{\frac{3}{6}}$, and $a^{\frac{2}{3}} \doteq a^{\frac{4}{6}}$.

**EXERCISE A.** Reduce $x^{\frac{1}{2}}$, $x^{\frac{2}{3}}$, $x^{\frac{3}{4}}$, $x^{\frac{4}{5}}$, and $x^{\frac{5}{6}}$ to surds of the same kind.

**NOTE.** After reducing the fractions to a com<sup>n</sup> den<sup>r</sup>, we may express the power and the root of each separately, as $a^{\frac{4}{6}} = (a^4)^{\frac{1}{6}}$. In this way we shall bring the quantities under the same fractional index.

**EXAMPLE 2.** Reduce $a^{\frac{5}{6}}$ and $b^{\frac{3}{4}}$ to a com<sup>n</sup> fractional exp<sup>t</sup>.

$$a^{\frac{5}{6}} \doteq a^{\frac{10}{12}} = (a^{10})^{\frac{1}{12}},$$
$$b^{\frac{3}{4}} \doteq b^{\frac{9}{12}} = (b^{9})^{\frac{1}{12}}.$$

**EXERCISE 1.** Reduce $x^{\frac{3}{4}}$ and $y^{\frac{1}{3}}$ to the same fractional exp<sup>t</sup>. Ans. $(x^9)^{\frac{1}{12}}$ and $(y^4)^{\frac{1}{12}}$.

2. The same with $3^{\frac{1}{3}}$ and $4^{\frac{1}{4}}$. Ans. $81^{\frac{1}{12}}$ and $64^{\frac{1}{12}}$.

B. The same with $c^{\frac{2}{3}}$, $c^{\frac{3}{4}}$, and $c^{\frac{5}{6}}$.

C. The same with $3^{\frac{1}{2}}$, $4^{\frac{1}{3}}$, and $5^{\frac{1}{4}}$.

D. The same with $(3x)^{\frac{1}{2}}$, $2y^{\frac{2}{3}}$, and $z^{\frac{3}{4}}$.

## PROBLEM III.

*To find such a Multiplier as shall render any given Surd rational.*

**RULE.** The surd being represented by $a^{\frac{m}{n}}$, and $p$ being any integer, the multiplier will be $a^{p-\frac{m}{n}}$ for $a^{\frac{m}{n}} \times a^{p-\frac{m}{n}} \doteq a^p$. For $p$ it is easiest to take the smallest integer $> \frac{m}{n}$.

**Note.** When a rational factor is attached to the surd (as $3a^{\frac{1}{3}}$) we take no notice of it.

**Exercise 1.** What multipliers will render the following surds rational, viz. $\sqrt{3}$ or $3^{\frac{1}{2}}$, $\sqrt[3]{5}$ or $5^{\frac{1}{3}}$, $3\sqrt[3]{6^2}$, and $ax^{\frac{2}{3}}$?

Ans. $3^{\frac{1}{2}}$ or $\sqrt{3}$, $5^{\frac{2}{3}}$ or $\sqrt[3]{5^2}$, $\sqrt[3]{6}$, and $x^{\frac{1}{3}}$.

A. What multipliers will render $4^{\frac{2}{3}}$, $5^{\frac{1}{4}}$, $3\sqrt[3]{7^2}$, and $x\sqrt[5]{y^7}$, rational?

## PROBLEM IV.

*To find such a Multiplier as shall render a Binomial or Trinomial Surd, in certain Cases, rational.*

**Rule.** When the surd is of any one of the forms contained in the first column of the following table, the multiplier will be expressed by the corresponding form in the second column; and, when the form of the given surd is contained in the second column, that of its multiplier will be found in the same line in the first column. The rational product will be found in the third column.

| Surd or Multiplier. | Multiplier or Surd. | Rational Product. |
|---|---|---|
| $a+\sqrt{b}$ | $a-\sqrt{b}$ | $a^2-b$ |
| $\sqrt{a}+b$ | $\sqrt{a}-b$ | $a-b^2$ |
| $\sqrt{a}+\sqrt{b}$ | $\sqrt{a}-\sqrt{b}$ | $a-b$ |
| $a+\sqrt[3]{b}$ | $a^2-a\sqrt[3]{b}+\sqrt[3]{b^2}$ | $a^3+b$ |
| $\sqrt[3]{a}+b$ | $\sqrt[3]{a^2}-\sqrt[3]{a}.b+b^2$ | $a+b^3$ |
| $a-\sqrt[3]{b}$ | $a^2+a\sqrt[3]{b}+\sqrt[3]{b^2}$ | $a^3-b$ |
| $\sqrt[3]{a}-b$ | $\sqrt[3]{a^2}+\sqrt[3]{a}.b+b^2$ | $a-b^3$ |
| $\sqrt[3]{a}+\sqrt[3]{b}$ | $\sqrt[3]{a^2}-\sqrt[3]{(ab)}+\sqrt[3]{b^2}$ | $a+b$ |
| $\sqrt[3]{a}-\sqrt[3]{b}$ | $\sqrt[3]{a^2}+\sqrt[3]{(ab)}+\sqrt[3]{b^2}$ | $a-b$ |
| $\sqrt[4]{a}+\sqrt[4]{b}$ | $(\sqrt[4]{a}-\sqrt[4]{b})(\sqrt{a}+\sqrt{b})$ | $a-b$ |
| $\sqrt[4]{a}-\sqrt[4]{b}$ | $(\sqrt[4]{a}+\sqrt[4]{b})(\sqrt{a}+\sqrt{b})$ | $a-b$ |

**Dem.** The proof will appear in each case from actual multiplication.

**Note.** The preceding table includes most cases that occur in practice; but the rule may be made more general; for, if a binomial surd is of the form $\sqrt[n]{a}\pm\sqrt[n]{b}$, it may be changed into the rational quantity $a\mp b$ by using as a multiplier $\sqrt[n]{(a^{n-1})}\mp\sqrt[n]{(a^{n-2}b)}+\sqrt[n]{(a^{n-3}b^2)}\dots\mp\sqrt[n]{(b^{n-1})}$; or, if we have a compound surd of the latter form, our multiplier must be a binomial of the former form. In like manner $\{a \pm \sqrt[n]{b}\} \times \{a^{n-1} \mp a^{n-2}\sqrt[n]{b} + a^{n-3}\sqrt[n]{b^2} \ldots. \mp \sqrt[n]{(b^{n-1})}\} = a^n \mp b$; and $\{\sqrt[n]{a} \pm b\} \times \{\sqrt[n]{(a^{n-1})} \mp \sqrt[n]{(a^{n-2})}b + \sqrt[n]{(a^{n-3})}b^2 \ldots. \mp b^{n-1} = a \mp b^n$. In all these cases the sign of the last term in the compound surd is + when that of the second term of the binomial is – ; but when the latter is +, the former is + or – according as $n$ is an odd or an even number.

EXERCISE 1. Find multipliers which shall render the following quantities rational, viz. $4 + \sqrt{5}$, $2^{\frac{1}{2}} - 3^{\frac{1}{2}}$, $x + y^{\frac{1}{3}}$, $4 + 2\sqrt[3]{3} + \sqrt[3]{9}$, $5^{\frac{1}{3}} - 6^{\frac{1}{3}}$, $\sqrt[3]{100} - \sqrt[3]{50} + \sqrt[3]{25}$, $\sqrt[4]{3} + \sqrt[4]{7}$, and $x^{\frac{1}{3}} - y^{\frac{2}{3}}$.

Ans. $4 - \sqrt{5}$, $2^{\frac{1}{2}} + 3^{\frac{1}{2}}$, $x^2 - xy^{\frac{1}{3}} + y^{\frac{2}{3}}$, $2 - \sqrt[3]{3}$, $25^{\frac{1}{3}} + 30^{\frac{1}{3}} + 36^{\frac{1}{3}}$, $\sqrt[3]{10} + \sqrt[3]{5}$, $\sqrt[4]{27} - \sqrt[4]{63} + \sqrt[4]{147} - \sqrt[4]{343}$, and $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}}$.

A. Determine what multipliers will render the following quantities rational, viz. $4 - 12^{\frac{1}{2}}$, $\sqrt{x} + \sqrt{y}$, $6 - \sqrt[3]{3}$, $9 - 3z^{\frac{1}{3}} + z^{\frac{2}{3}}$, $5^{\frac{1}{3}} + 6^{\frac{1}{3}}$, $36^{\frac{1}{3}} + 42^{\frac{1}{3}} + 49^{\frac{1}{3}}$, $5^{\frac{1}{4}} - 3^{\frac{1}{4}}$, and $x^{\frac{2}{3}} + y^{\frac{2}{3}}$.

COR. 1. The trinomial surd, $\sqrt{a} + \sqrt{b} + \sqrt{c}$, may be made rational by two mult$^{ns}$. If multiplied by $\sqrt{a} + \sqrt{b} - \sqrt{c}$, it becomes $a + 2\sqrt{(ab)} + b - c$; and, if this is again mult$^d$ by $a - 2\sqrt{(ab)} + b - c$, it becomes $(a + b - c)^2 - 4ab$. The process is precisely similar if the signs of any of the terms of the given trinomial are negative.

COR. 2. In some peculiar cases a multiplier may be found which shall render any root of a binomial or trinomial surd rational. Thus $\sqrt[3]{\{\sqrt{a^3} + \sqrt{(a^3 - b^3)}\}}$ becomes $b$, if mult$^d$ by $\sqrt[3]{\{\sqrt{a^3} - \sqrt{(a^3 - b^3)}\}}$.

EXERCISE B. Find a multiplier which shall render $\sqrt[3]{\left\{\frac{b}{2} - \sqrt{\left(\frac{b^2}{4} + \frac{a^3}{27}\right)}\right\}}$ rational.

## PROBLEM V.

*To change a Fraction whose Denominator is either a simple Surd, or, in certain Cases, a compound Surd, into an equal Fraction with a rational Denominator.*

RULE. By one of the two preceding problems find such a multiplier, when possible, as shall render the denominator rational, and by that multiply both terms of the fraction.

When the denominator becomes 1, the fraction becomes an integer.

**Example.** Reduce $\frac{2x}{3y^{\frac{1}{3}}}$ and $\frac{\sqrt[3]{a}+\sqrt[3]{b}}{\sqrt[3]{a}-\sqrt[3]{b}}$ to fractions having rational denominators.

The respective multipliers are $y^{\frac{2}{3}}$ and $\sqrt[3]{a^2}+\sqrt[3]{(ab)}+\sqrt[3]{b^2}$; and the terms of the two fractions being multiplied severally by these, give us $\frac{2xy^{\frac{2}{3}}}{3y}$ and $\frac{a+2\sqrt[3]{(a^2b)}+2\sqrt[3]{(ab^2)}+b}{a-b}$.

**Note.** Such a transformation of a fraction is often required for various objects, but especially for facilitating numerical computation. Thus, if we were about to express the fraction $\frac{2}{\sqrt{3}}$ decimally, we should be obliged, if we took the fraction as it stands, to extract the square root of 3, which is an interminate decimal, and then to divide 2 by the quotient. If we bring the numerator under the radical sign, changing the fraction into $\sqrt{\frac{4}{3}}=\sqrt{1\cdot\dot{3}}$, we have the numerical operation shortened considerably, and, in fact, as much so as it can be, if we have actually to extract the square root. If, however, we have the advantage of a table of roots, we cannot use it advantageously to obtain the root of $1\cdot\dot{3}$; but if we reduce the fraction, by this rule, into $\frac{2}{3}\sqrt{3}$, we find at once, from the table, $\sqrt{3}=1\cdot73205081$, and two thirds of that is $1\cdot15470054$. For this reason no algebraical formula, intended for numerical application, is left with a surd divisor, if it can be avoided.

**Exercise** 1. Reduce the following fractions to fractions with rational den$^{rs}$, or to integers, viz. $\frac{a}{\sqrt[3]{b}}$, $\frac{8}{\sqrt{3}}$, $\frac{5y}{2x^{\frac{1}{2}}}$, $\frac{4}{\sqrt[5]{5^2}}$, and $5x^{\frac{1}{2}}y^{-\frac{1}{2}}$.

Ans. $\frac{a}{b}\sqrt[3]{b^2}$, $\frac{8}{3}\sqrt{3}$, $\frac{5x^{\frac{1}{2}}y}{2x}$, $\frac{4}{5}\sqrt[5]{125}$, and $5x^{\frac{1}{2}}y^{\frac{1}{2}}y^{-1}$.

A. The same with $\frac{x}{\sqrt{a}}$, $\frac{3}{\sqrt[3]{4}}$, $\frac{b}{2\sqrt[3]{y^2}}$, and $a^{\frac{1}{2}}y^{-\frac{2}{3}}$.

2. The same with $\frac{x}{x-\sqrt{y}}$, $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$, $\frac{21}{\sqrt{10}-\sqrt{3}}$, $\frac{\sqrt[3]{5}-\sqrt[3]{4}}{\sqrt[3]{5}+\sqrt[3]{4}}$, and $\frac{1}{3^{\frac{3}{2}}-2^{\frac{3}{2}}}$.

Ans. $\frac{x^2+x\sqrt{y}}{x^2-y}$, $\frac{a-2\sqrt{(ab)}+b}{a-b}$, $3(\sqrt{10}+\sqrt{3})$, $\frac{1}{9}(1-2\sqrt[3]{100}+2\sqrt[3]{80})$, and $\frac{1}{19}(3\sqrt{3}+2\sqrt{2})$.

B. The same with $\frac{a}{a+\sqrt{x}}$, $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$, $\frac{1}{\sqrt{10}-\sqrt{9}}$, $\frac{10}{2-\sqrt[3]{6}}$, and $\frac{3^{\frac{1}{2}}-2^{\frac{1}{2}}}{3^{\frac{1}{2}}+2^{\frac{1}{2}}}$.

3. Divide $\sqrt{20}-\sqrt{12}$ by $\sqrt{5}+\sqrt{3}$.
Ans. $8-2\sqrt{15}$.

C. Divide $\sqrt{28}+\sqrt{20}$ by $\sqrt{7}-\sqrt{5}$.

## PROBLEM VI.

### *To add any Number of Surds together, or to subtract one Surd from another.*

RULE. If the surds are of the same kind, reduce them to their simplest form, by Pr. 1 of Ch. XVI. Then, if the surd factor is the same in all, add or subtract its rational coefficients. But if the surd factors are of different kinds, or if the surd parts, when reduced to their simplest form, are not the same in all, the addition or subtraction can be performed only by connecting together the given quantities with the sign + or −.

EXAMPLE. What are the sum and difference of $\sqrt{50}$ and $\sqrt{18}$?

$$\sqrt{50}+\sqrt{18} \doteq 5\sqrt{2}+3\sqrt{2}=8\sqrt{2}.$$
$$\sqrt{50}-\sqrt{18} \doteq 5\sqrt{2}-3\sqrt{2}=2\sqrt{2}.$$

EXERCISE 1. What are the sum and difference of $\sqrt[3]{54}$ and $\sqrt[3]{128}$? Ans. $7\sqrt[3]{2}$ and $\sqrt[3]{2}$.

A. Add together $\sqrt{80}$, $\sqrt{125}$, and $\sqrt{245}$.

B. From $\sqrt{175}$ subtract $\sqrt{63}$.

C. What are the sum and dif$^{ce}$ of $\sqrt[3]{500}$ and $\sqrt[3]{108}$?

D. From $\sqrt{(m^2n^4x)}$ subtract $\sqrt{(p^4q^2x)}$.

2. Find the sum and dif$^{ce}$ of $6\sqrt{\frac{3}{4}}$ and $3\sqrt{\frac{4}{3}}$.
Ans. $5\sqrt{3}$ and $\sqrt{3}$.

E. Find the sum and dif$^{ce}$ of $3\sqrt{\frac{2}{5}}$ and $4\sqrt{\frac{1}{10}}$.

3. What are the sum and dif$^{ce}$ of $2\sqrt[3]{\frac{1}{5}}$ and $3\sqrt[3]{\frac{1}{40}}$?
Ans. $\frac{7}{10}\sqrt[3]{25}$ and $\frac{1}{10}\sqrt[3]{25}$.

F. From $3\sqrt{5}$ subtract $2\sqrt{4}$.

## PROBLEM VII.

### *To multiply Surds together, or to divide one Surd by another.*

RULE. *When the surds are of the same kind,* the rules have already been given in Prˢ 2 and 3 of Ch. XVI. *When they are of different kinds*, they must be reduced to the same fractional index, by Pr. 2 of this chapter, and, being *then* of the same kind, they may be treated accordingly. The products or quotients of fractional powers *of the same* quantity may be found by taking the sum or difference of their exponents, according to Thˢ 2 and 3 of Ch. XXXI.

EXAMPLE 1. What are the product and quotient of $3^{\frac{1}{3}}$ and $2^{\frac{1}{2}}$?

$$3^{\frac{1}{3}} \doteq 3^{\frac{2}{6}} = (3^2)^{\frac{1}{6}} = \sqrt[6]{9},$$
$$2^{\frac{1}{2}} \doteq 2^{\frac{3}{6}} = (2^3)^{\frac{1}{6}} = \sqrt[6]{8}.$$

The product is $\sqrt[6]{72}$, and the quotient of the former divided by the latter is $\sqrt[6]{1\frac{1}{8}}$.

NOTE. Rational coefficients must be multiplied or divided by each other, and the product or quotient multiplied into that of the surd factors.

EXAMPLE 2. What are the product and quotient of $4a^{\frac{3}{4}}$ by $3a^{\frac{2}{3}}$?

$$4a^{\frac{3}{4}} \times 3a^{\frac{2}{3}} \doteq 12a^{\frac{3}{4}+\frac{2}{3}} = 12a^{\frac{17}{12}}.$$
$$4a^{\frac{3}{4}} \div 3a^{\frac{2}{3}} = \tfrac{4}{3}a^{\frac{3}{4}-\frac{2}{3}} = \tfrac{4}{3}a^{\frac{1}{12}}.$$

EXERCISE 1. Multiply and divide $12x^{\frac{2}{3}}$ by $3x^{\frac{1}{2}}$.

Ans. $36x^{\frac{7}{6}}$ and $4x^{\frac{1}{6}}$.

2. Multiply and divide $\sqrt[3]{x^2}$ by $\sqrt{y}$.

Ans. $\sqrt[6]{(x^4y^3)}$ and $\sqrt[6]{(x^4y^{-3})}$.

A. Required the product and quotient of $2\sqrt[3]{z^2}$ by $3\sqrt{z^3}$.

B. The same of $\frac{2}{3}a\sqrt[3]{b^2}$ by $\frac{3}{4}b\sqrt{a^3}$.

3. Express as one fraction $5^{\frac{3}{4}} \times 4^{\frac{2}{3}} \times 3^{\frac{3}{2}} \div 60^{\frac{5}{6}}$.

Ans. $\sqrt[12]{\frac{6561}{80}}$.

C. Divide $(\frac{3}{2})^{\frac{3}{4}}$ by $(\frac{3}{4})^{\frac{2}{3}}$.

## THEOREM I.

*The Sum or Difference of a Rational Quantity and a Quadratic Surd cannot be a Rational Quantity.*

DEM. If possible let $a \pm \sqrt{b}$ be $= c$. Then $\pm\sqrt{b} \doteq c - a$, or an irrat¹ qᵗʸ = a rat¹, which, by the defⁿ of the former, is impossible.

### THEOREM II.

*The Sum or Difference of a Rational Quantity and a Quadratic Surd cannot be a Quadratic Surd.*

DEM. If possible let $a \pm \surd b$ be $= \surd c$. Then, squaring both sides, we have $a^2 \pm 2a \surd b + b = c$, and $\therefore \pm \surd b = (a^2 + b - c) \div 2a$, or a surd = a rat$^{l}$ q$^{ty}$, which is impossible.

NOTE. In this and the other dem$^{s}$ it is understood that $\surd a$, $\surd b$, &c. denote real surds and not rat$^{l}$ q$^{ties}$ under that form. In this it is also understood that $a$ is a real q$^{ty}$; for, if $a$ were $= 0$, then $a \pm \surd b$ might be $= \surd c$, $b$ being $= c$.

COR. 1. The sum or difference of two q$^{c}$ surds cannot be a rat$^{l}$ quantity. For, if $\surd a \pm \surd b \doteq c$, then $c \mp \surd b \doteq \surd a$, which, by the theorem, is impossible.

COR. 2. Therefore, $\surd a$ being irrat$^{l}$ and $n$ rat$^{l}$, $n \surd a$ is irrat$^{l}$.

### THEOREM III.

*If the Sum or Difference of a Rational Quantity and a Quadratic Surd is equal to the sum or difference of a Rational Quantity and a Quadratic Surd, the Rational Quantity is equal to the Rational and the Surd to the Surd.*

DEM. Let $a \pm \surd b$ be $= c \pm \surd d$; then $a \doteq c$ and $b = d$. For, if $a$ be not $= c$, let it be $= c \pm e$. Then, since $c \pm e \pm \surd b \doteq c \pm \surd d$, we have $\pm e \pm \surd b = \pm \surd d$, which, by Th. 2, is impossible, for $e$ must be rat$^{l}$, being $= \pm a \mp c$. $\therefore a \doteq c$; and, taking these equals from the given equation, we have $\pm \surd b = \pm \surd d$, and $b = d$.

If for $\surd b$ and $\surd d$ we substitute $m \surd b$ and $n \surd d$, the dem$^{n}$ is the same.

### THEOREM IV.

*If two Quadratic Surds have the same Irrational Part, their Product and Quotient are Rational.*

DEM. Let $\surd a$ and $\surd b$ have the same irrat$^{l}$ part, viz. $\surd c$, and let $\surd a$ be $= m \surd c$ and $\surd b = n \surd c$: then $\surd a \surd b \doteq mnc$, which is rat$^{l}$.

Again, $\frac{\sqrt{a}}{\sqrt{b}} \doteq \frac{m\sqrt{c}}{n\sqrt{c}} = \frac{m}{n}$, which is also rat$^{l}$.

COR. 2. If $\sqrt{a}\sqrt{b}$ or $\sqrt{a} \div \sqrt{b}$ is a surd, then $\sqrt{a}$ and $\sqrt{b}$ have not the same surd part; for if they had, their product and quotient would be rat$^{l}$.

### THEOREM V.

*If the Product or Quotient of two Quadratic Surds is a Rational Quantity, the two Surds have the same Irrational Part.*

DEM. Let $\sqrt{a}\sqrt{b}$ be $= c$, $c$ being rational: then, putting $n$ for $c \div a$, or $na$ for $c$, $\sqrt{a}\sqrt{b} \doteq na$, or $\sqrt{b} \doteq n\sqrt{a}$: That is, $\sqrt{a}$ and $\sqrt{b}$ have the same irrat$^{l}$ part. Next let $\sqrt{a} \div \sqrt{b}$ be $= c$: then, multiplying both by $\sqrt{b}$, we have $\sqrt{a} = c\sqrt{b}$, and again $\sqrt{a}$ and $\sqrt{b}$ have the same irrat$^{l}$ part.

COR. 1. If $m\sqrt{a} \doteq n\sqrt{b}$, then $\sqrt{a}$ and $\sqrt{b}$ must contain the same surd part; for $\sqrt{a} \div \sqrt{b} \doteq n \div m$, which is rat$^{l}$.

COR. 2. If two q$^{e}$ surds cannot be reduced to others which have the same irrat$^{l}$ part, their product and quotient are irrat$^{l}$.

COR. 3. From this and Theorem IV. it follows that if $\sqrt{a}\sqrt{b} \doteq \sqrt{x}\sqrt{y}$, then, according as $\sqrt{a}$ and $\sqrt{b}$ have or have not the same irrational part, so $\sqrt{x}$ and $\sqrt{y}$ have or have not.

### THEOREM VI.

*The Sum or Difference of two Quadratic Surds, which have not the same Irrational Part, cannot be a Quadratic Surd.*

DEM. Let $\sqrt{a} \pm \sqrt{b}$ be $= \sqrt{c}$, squaring both sides, $a \pm 2\sqrt{a}\sqrt{b} + b \doteq c$; or $\pm\sqrt{a}\sqrt{b} \doteq \frac{1}{2}(c - a - b)$; which (by Th. 5, Cor. 2) is impossible if $\sqrt{a}$ and $\sqrt{b}$ have not the same rat$^{l}$ part.

### THEOREM VII.

*If* $\sqrt{\mathrm{a}} \pm \sqrt{\mathrm{b}} \doteq \sqrt{\mathrm{x}} \pm \sqrt{\mathrm{y}}$; *then, according as* $\sqrt{\mathrm{a}}$ *has or has not the same surd part with* $\sqrt{\mathrm{b}}$, *so* $\sqrt{\mathrm{x}}$ *has or has not the same surd part with* $\sqrt{\mathrm{y}}$.

DEM. If $\sqrt{a}$ has the same surd part with $\sqrt{b}$, let that part be $\sqrt{c}$; and let $\sqrt{a}$ be $= m\sqrt{c}$ and $\sqrt{b} = n\sqrt{c}$. Then $(m \pm n)\sqrt{c} \doteq \sqrt{a} \pm \sqrt{b} = \sqrt{x} \pm \sqrt{y}$, which cannot be unless $\sqrt{x}$ and $\sqrt{y}$ have the same surd part (Th. 6).

But if $\sqrt{a}$ and $\sqrt{b}$ have not the same surd part, neither have $\sqrt{x}$ and $\sqrt{y}$. For if $\sqrt{x}$ and $\sqrt{y}$ had, $\sqrt{a}$ and $\sqrt{b}$ would also have, as has just been proved.

## THEOREM VIII.

*If* $\sqrt{\mathrm{a}} + \sqrt{\mathrm{b}}$ *be* $= \sqrt{\mathrm{x}} + \sqrt{\mathrm{y}}$, *and if* $\sqrt{\mathrm{a}}$ *and* $\sqrt{\mathrm{b}}$ *have not the same surd part, then* $\sqrt{\mathrm{a}} \doteq \sqrt{\mathrm{x}}$ *and* $\sqrt{\mathrm{b}} = \sqrt{\mathrm{y}}$, or $\sqrt{\mathrm{a}} = \sqrt{\mathrm{y}}$ *and* $\sqrt{\mathrm{b}} = \sqrt{\mathrm{x}}$.

**DEM.** Squaring both sides of the given equation,

$$a + 2\sqrt{(ab)} + b \doteq x + 2\sqrt{(xy)} + y.$$

Then, since $\sqrt{(ab)}$ is irrat$^{l}$ (Th. 5, Cor. 2), and conseq$^{y}$ $\sqrt{(xy)}$ (Th. 1). $\therefore a + b \doteq x + y$, and $2\sqrt{(ab)} = 2\sqrt{(xy)}$ (Th. 3). Taking equals from equals,

$$a - 2\sqrt{ab} + b \doteq x - 2\sqrt{xy} + y,$$
$$\text{and } \sqrt{a} - \sqrt{b} \doteq \pm(\sqrt{x} - \sqrt{y}).$$
$$\text{But } \sqrt{a} + \sqrt{b} \doteq + \sqrt{x} + \sqrt{y}.$$
$$\therefore 2\sqrt{a} = 2\sqrt{x} \text{ or } 2\sqrt{y}, \text{ and } 2\sqrt{b} = 2\sqrt{y} \text{ or } 2\sqrt{x}.$$

**COR.** We have a similar result if $\sqrt{a} - \sqrt{b} \doteq \sqrt{x} - \sqrt{y}$, under the same circumstances.

## THEOREM IX.

*If* $\sqrt{(\mathrm{a} + \sqrt{\mathrm{b}})} \doteq \mathrm{c} + \mathrm{d}$ (c *and* d *being, one or both, Quadratic Surds not containing the same Irrational Part*), *then will* $\sqrt{(\mathrm{a} - \sqrt{\mathrm{b}})}$ *be* $= \mathrm{c} - \mathrm{d}$.

**DEM.** Squaring both sides of the given equation,

$$a + \sqrt{b} \doteq c^2 + 2cd + d^2.$$

But (Th. 2. Cor. 2, or Th. 5, Cor. 2) $cd$ is irrat$^{l}$: $\therefore$ (Th. 3) $a \doteq c^2 + d^2$ and $\sqrt{b} = 2cd$.

Subtracting the latter of these two eq$^{ns}$ from the former, we have $a - \sqrt{b} = c^2 - 2cd + d^2$, and conseq$^{y}$ $\sqrt{(a - \sqrt{b})} = c - d$.

**COR.** In like manner we prove the converse.

**NOTE.** In a somewhat similar manner we might prove

**that if $\sqrt{(\sqrt{a}+\sqrt{b})}\doteq c+d$, under the same conditions, then $\sqrt{(\sqrt{a}-\sqrt{b})}\doteq c-d$; but we should prove at the same time that neither equation could hold true unless either $\sqrt{a}$ or $\sqrt{b}$ were rational, and the case would then be the same as that in the theorem.**

### THEOREM X.

*If* $\sqrt[3]{(\mathrm{a}+\sqrt{\mathrm{b}})}\doteq \mathrm{c}+\sqrt{\mathrm{d}}$ (a *and* c *being Rational*), *then* $\sqrt[3]{(a-\sqrt{b})}\doteq c-\sqrt{d}$.

DEM. Cubing both sides of the given eq$^{n}$, we have $a+\sqrt{b}=c^3+3c^2\sqrt{d}+3cd+d\sqrt{d}=c^3+3cd+(3c^2+d)\sqrt{d}$. Now, $\sqrt{d}$ being irrat$^{l}$, $(3c^2+d)\sqrt{d}$ is irrat$^{l}$ (Th. 2, Cor. 2). $\therefore$ (Th. 3) $a\doteq c^3+3cd$, and $\sqrt{b}=(3c^2+d)\sqrt{d}$. Subtracting the latter of these two eq$^{ns}$ from the former, $a-\sqrt{b}\doteq c^3+3cd-(3c^2+d)\sqrt{d}=c^3-3c^2\sqrt{d}+3cd-d\sqrt{d}=(c-\sqrt{d})^3$; and $\therefore \sqrt[3]{(a-\sqrt{b})}\doteq c-\sqrt{d}$.

COR. In like manner we prove the converse.

### THEOREM XI.

*If* $\sqrt[3]{(\sqrt{\mathrm{a}}+\sqrt{\mathrm{b}})}\doteq\sqrt{\mathrm{c}}+\sqrt{\mathrm{d}}$ (a *and* b *not having the same Surd Part*), *then* $\sqrt[3]{(\sqrt{\mathrm{a}}\sim\sqrt{\mathrm{b}})}=\sqrt{c}-\sqrt{d}$.

DEM. Cubing both sides of the given eq$^{n}$,

$$\sqrt{a}+\sqrt{b}\doteq(c+3d)\sqrt{c}+(3c+d)\sqrt{d}.$$

Then (Th. 8) $\sqrt{a}\doteq(c+3d)\sqrt{c}$, and $\sqrt{b}=(3c+d)\sqrt{d}$,
or $\sqrt{b}\doteq(c+3d)\sqrt{c}$, and $\sqrt{a}=(3c+d)\sqrt{d}$.

In either case take the latter eq$^{n}$ from the former.

Then $\sqrt{a}\sim\sqrt{b}\doteq(c+3d)\sqrt{c}-(3c+d)\sqrt{d}$,
and $\therefore \sqrt[3]{(\sqrt{a}\sim\sqrt{b})}=\sqrt{c}-\sqrt{d}$.

COR. In the same manner we prove the converse.

### PROBLEM VIII.

*Having given a Binomial Surd, one of whose Terms is Rational and the other a Quadratic Surd, to obtain its Square Root in the Form of a Binomial Quadratic Surd in all Cases in which it is possible.*

RULE. Expressing the binomial surd generally as

$a \pm b\sqrt{c}$, find two quantities, $g$ and $h$, such that their sum shall be $= a$, and twice the square root of their product $= b\sqrt{c}$. Then we shall have $a \pm b\sqrt{c} = g \pm 2\sqrt{(gh)} + h$, and conseq$^{y}$ $\sqrt{(a \pm b\sqrt{c})} = \sqrt{g} \pm \sqrt{h}$.

**Dem.** The rule contains its own demonstration, provided that two quantities can be found to fulfil the conditions supposed. That is sometimes possible, but in other cases not so. To determine *when* it is possible, we have only to find the values of $\sqrt{g}$ and $\sqrt{h}$ from the two equations $g + h = a$ and $2\sqrt{(gh)} = b\sqrt{c}$, viz.

$$\sqrt{g} = \tfrac{1}{2}\sqrt{(a + b\sqrt{c})} + \tfrac{1}{2}\sqrt{(a - b\sqrt{c})}, \text{ and}$$
$$\sqrt{h} = \tfrac{1}{2}\sqrt{(a + b\sqrt{c})} - \tfrac{1}{2}\sqrt{(a - b\sqrt{c})}.$$

From these we see that ($a$, $b$, and $\sqrt{c}$ being real quantities) $\sqrt{g}$ and $\sqrt{h}$ have always real values when $\sqrt{(a + b\sqrt{c})}$ and $\sqrt{(a - b\sqrt{c})}$ are real quantities; that is, when $a$ is positive and $> b\sqrt{c}$, which we suppose positive. But if $a$ is positive and not $> b\sqrt{c}$, or if $a$ is negative, $a - b\sqrt{c}$ will be negative and its square root impossible, and conseq$^{y}$ the values of $\sqrt{g}$ and $\sqrt{h}$ both impossible.

By squaring the two last equations, we find $g = \frac{1}{2}\{a + \sqrt{(a^2 - b^2c)}\}$ and $h = \frac{1}{2}\{a - \sqrt{(a^2 - b^2c)}\}$, which will always be *rational* quantities when $a^2 - b^2c$ is a square; and, when that is the case, either the value of $g$ or that of $h$ *may be* a square, and $\sqrt{g}$ or $\sqrt{h}$ rational; but *both* cannot be so, for then $g \pm 2\sqrt{(gh)} + h$ would be rational, and conseq$^{y}$ $a \pm b\sqrt{c}$ rational, which, by our hypothesis, it is not.

**Note 1.** We have the converse of this problem in Cor. 4 of Pr. 1.

**Example.** Find the square root of $5 + 2\sqrt{6}$.

Here we have $g + h = 5$ and $2\sqrt{(gh)} = 2\sqrt{6}$. Squaring each of these equations and taking the difference of the squares, we have $g^2 - 2gh + h^2 = 1$, and $\therefore g - h = 1$. $\therefore$ $g = 3$ and $h = 2$, and $\sqrt{(5 + 2\sqrt{6})} = \sqrt{3} + \sqrt{2}$.

It is better to work out this process in each case than to trust to the recollection of the formulæ for the values of $g$ and $h$.

**Exercise 1.** Find the square roots of $7 + 4\sqrt{3}$, $17 - 6\sqrt{8}$, and $12 - 2\sqrt{35}$.

Ans. $2 + \sqrt{3}$, $3 - \sqrt{8}$, and $\sqrt{7} - \sqrt{5}$.

**A.** Express, as binomial surds, the square roots of $37 - 12\sqrt{7}$, $1 - 2\sqrt{3}$, and $3\frac{1}{2} - 2\sqrt{\frac{3}{2}}$.

**Note 2.** When the given binomial contains a quadratic factor, the process will be facilitated by taking the root of that factor separately. Thus $\surd(45-27\surd 2) \doteq 3\surd(5-3\surd 2)$.

EXERCISE 2. Extract the square root of $20-8\surd 6$.
Ans. $2(\surd 3-\surd 2)$ or $\surd 12-\surd 8$.

B. The same with $120+40\surd 5$ and $100-25\surd 15$.

NOTE 3. The preceding example and exercises have been limited to the case of numerical quantities. When the given binomial consists partially or altogether of quantities expressed by letters, as $a \pm b\surd c$, its root, being then of the form

$$\surd\{\tfrac{1}{2}a+\tfrac{1}{2}\surd(a^2-b^2c)\} \pm \surd\{\tfrac{1}{2}a-\tfrac{1}{2}\surd(a^2-b^2c),$$

is too complicated to be of any use, unless in the case in which the rational term of the given binomial surd is itself a binomial, and such as to render the whole a complete square, as $(x+y)+2\surd(xy)$. In fact this must also be the character of those numerical binomial surds whose roots can be obtained by this rule: they are really trinomials under the appearance of binomials. Thus $\surd(5+2\surd 6)$ is really $\surd\{(3+2)+2\surd 6\} = \surd 3+\surd 2$. Similar cases may sometimes occur with literal quantities, as in the following exercises, which are resolved by the general rule.

EXERCISE 3. Find the square root of $2a-2\surd(a^2-1)$.
Ans. $\surd(a+1)-\surd(a-1)$.

C. The same with $4x-2\surd(4x^2-9y^2)$.

DEF. 2. Although no even root of a negative quantity can be extracted, or have any real existence, yet such a root may be *expressed* as a surd, as $\surd-3$. It is then called an *Impossible* or *Imaginary Quantity*, a term which is also applied to every expression in which such a root is contained, unless by any process the impossible root can be eliminated, or, in other words, unless the impossible parts destroy each other. Quantities represented by expressions in which no impossible root is necessarily involved, are called *Real Quantities.*

## PROBLEM IX.

*Having given a Binomial Quadratic Surd, to find its Cube Root in the Form of another Binomial Quadratic Surd, in all Cases in which it is possible, and to determine when it is possible.*

It may have been observed from some of the last exercises in Pr. 1, that the cube root of a quad$^e$ surd is sometimes a quad$^e$ surd. Other instances occurred in Pr. 3 of Ch. xxx.; and, in Note 4 to that problem, it was promised that the general principle, on which such a result depends, would be more fully developed, and the conditions under which it is possible defined, in a subsequent chapter. That desideratum this problem is intended to supply.

NOTE. When the terms of our given binomial are both surds, it is understood that they have not the same surd part, otherwise the binomial might then be expressed as $(a+b)\sqrt{x}$, which cannot be made more simple.

(1.) We know that $(\sqrt{g} \pm \sqrt{h})^3 =$

$$\sqrt{g^3} \pm 3\sqrt{(g^2h)} + 3\sqrt{(gh^2)} \pm \sqrt{h^3} =$$
$$g\sqrt{g} \pm 3g\sqrt{h} + 3h\sqrt{g} \pm h\sqrt{h} =$$
$$(g+3h)\sqrt{g} \pm (3g+h)\sqrt{h}.$$

If, ∴, we have given any binomial surd of the form $a\sqrt{g} \pm b\sqrt{h}$, its cube root will be $\sqrt{g} \pm \sqrt{h}$, provided that $a$ be $= g + 3h$, and $b = 3g + h$.

EXAMPLE 1. Find the cube root of $21\sqrt{6} - 23\sqrt{5}$, in the form of a binomial quadratic surd.[*]

Here $g = 6$ and $h = 5$. Then $a = 21 = 6 + 3.5$, and $b = 23 = 5 + 3.6$. ∴ the required root is $\sqrt{6} - \sqrt{5}$.

EXERCISE 1. Determine the cube roots of $14\sqrt{5} - 18\sqrt{3}$ and $9\sqrt{3} + 11\sqrt{2}$. Ans. $\sqrt{5} - \sqrt{3}$, and $\sqrt{3} - \sqrt{2}$.

A. The same with the cube roots of $16\sqrt{7} + 24\sqrt{3}$ and $43\sqrt{13} - 49\sqrt{10}$.

(2.) We are not however to infer that the given binomial, $a\sqrt{g} \pm b\sqrt{h}$, has no square root of the form required although $a$ may not be $= g + 3h$ nor $b = 3g + h$. We merely know that the root will not be $\sqrt{g} \pm \sqrt{h}$. For, if any number $n$ can be found such that $na = g + 3h$, and $nb = 3g + h$, we can then change the cube root of the given binomial into $\sqrt[3]{\frac{1}{n}}\sqrt[3]{(na\sqrt{g} \pm nb\sqrt{h})}$, which, as appears from the previous investigation, is $\sqrt[3]{\frac{1}{n}} \times (g \pm \sqrt{h})$. If $n$ is an exact cube, we may put $m$ for $\sqrt[3]{\frac{1}{n}}$, and the required root will be $m\sqrt{g} \pm m\sqrt{h}$. When $n$ is not an exact cube we have no *proper* solution of our question, although we have a solution of a different kind which may sometimes facilitate numerical computation. To find $n$, we have merely to divide $g + 3h$ by $a$, or $3g + h$ by $b$.

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[*] This and the other examples, with several of the exercises which follow, are taken from some of the best of our standard treatises on Algebra, especially those which enter most carefully into this branch of the subject, in order to show the comparative simplicity of the methods here given.

EXAMPLE 2. Required the cube root of $2\sqrt{7} + 3\sqrt{3}$.

Here $n \doteq \frac{7 + 9}{2}$ or $\frac{21 + 3}{3}$; and since the two quotients are equal, being each 8, the required root is $\frac{1}{2}(\sqrt{7} + \sqrt{3})$.

EXERCISE 2. Find a simpler expression for $\sqrt[3]{(1125\sqrt{3} - 1375\sqrt{2})}$. Ans. $5\sqrt{3} - 5\sqrt{2}$.

B. The same for $\sqrt[3]{(6\sqrt{15} - 7\sqrt{11})}$ and $\sqrt[3]{(88\sqrt{2} + 72\sqrt{3})}$.

(3.) When the solution is possible, our two quotients would always come out equal, if, in the given binomial, the factors of one or both terms were not distributed differently from their arrangement in the expression formed by directly cubing the root; and, in fact, we should always have our quotients not only equal, but each $= 1$, provided that our factors were properly arranged in the given binomial. For, whenever the operation is possible, the root may be put into the form $\sqrt{x} \pm \sqrt{y}$, the cube of which is $(x + 3y)\sqrt{x} + (3x + y)\sqrt{y}$; and, if that is $= a\sqrt{g} + b\sqrt{h}$, then (by Th. 8), since $\sqrt{g}$ and $\sqrt{h}$ by hypothesis have not the same surd part, $(x + 3y)\sqrt{x} \doteq a\sqrt{g}$ and $(3x + y)\sqrt{y} = b\sqrt{h}$, the one of the two equals being always convertible into the other by a re-arrangement of the factors. Thus, if we have given $7\sqrt{20} + 6\sqrt{27}$, in which the quotients are not equal, we may change it into $14\sqrt{5} + 18\sqrt{3}$, or $7\sqrt{20} + 9\sqrt{12}$, in either of which the quotients are equal.

EXERCISE 3. What is the cube root of $5\sqrt{175} - 9\sqrt{54}$? Ans. $\sqrt{7} - \sqrt{6}$.

C. The same of $8\sqrt{27} + 8\sqrt{28}$.

(4.) If, in our given binomial, we have simply $\sqrt{g}$ or $\sqrt{h}$, instead of $a\sqrt{g}$ or $b\sqrt{h}$, we may take 1 for $a$ or $b$; or, if $g$ or $h$ contain a square factor, *its* root may be taken as $a$ or $b$. Thus, if we have given $\sqrt{12} + \sqrt{18}$, it may be written $1\sqrt{12} + 1\sqrt{18}$, or $2\sqrt{3} + 3\sqrt{2}$. We must try to separate our factors so that we may have $a$ or $na = g + 3h$ and $b$ or $nb = 3g + h$.

(5.) When our given binomial is of the form $c + b\sqrt{h}$, we may take 1 for $g$ and $c$ for $a$; or we may take 1, or any factor of $c$, for $a$, and $(c \div a)^2$ for $g$, and proceed as before.

**Example 3.** Express $\sqrt[3]{(10+\sqrt{108})}$ in its simplest form.

Taking the given binomial as $10\sqrt{1}+6\sqrt{3}$, we find $10=1+9$ and $6=3+3$. $\therefore$, by (1), our root is $1+\sqrt{3}$.

**Exercise 4.** Find $\sqrt[3]{(7-5\sqrt{2})}$ and $\sqrt[3]{(38+17\sqrt{5})}$.
Ans. $1-\sqrt{2}$ and $2+\sqrt{5}$.

D. Express more simply the cube roots of $16+8\sqrt{5}$, $10\sqrt{7}+22$, $50-19\sqrt{7}$, and $72-32\sqrt{5}$.

**Example 4.** Required $\sqrt[3]{(11+5\sqrt{7})}$.

Writing this $11\sqrt{1}+5\sqrt{7}$ and taking $n=(1+21)\div 11$ or $(3+7)\div 5$, since the two quotients are equal, the required root is $(\sqrt{1}+\sqrt{7})\div\sqrt[3]{2}$. But, since $\sqrt[3]{2}$ is not rat', we have not a *true* solution, and, as we shall see presently, we cannot have.

(6.) By the preceding method all cases may be resolved that are capable of solution, if we distribute our factors properly; but the trouble of doing this is sometimes very great, especially when the terms of the root are fractional, —a case which is quite admissible. By the same method also cubic equations may be resolved in particular instances of what is called the *irreducible case* (Ch. XXX. Note 3), and indeed in all instances in which any one of the three values of the unknown quantity is an integer, or even a fraction of finite terms; but the trouble of arranging the factors will generally be greater than that of finding the values of the unknown quantity by trials.

Thus, if we have given $x^3-9x^2+20x=12$; putting $x=y+3$, we obtain $y^3-7y=6$. Then again making $y=z+\frac{7}{3z}$, we have $z^6-6z^3=-\frac{343}{27}$. Hence we find $z^3=3\pm 10\sqrt{(-\frac{1}{27})}=-1\times(-3\sqrt{1}\mp\frac{5}{3}\sqrt{(-\frac{4}{3})}$, and hence, by our rule, $z=-1\pm\sqrt{(-\frac{4}{3})}$ and $\frac{7}{3z}=-1\mp\sqrt{(-\frac{4}{3})}$ (Ch. XXX. Note 1). $\therefore y=-2$ and $x=1$. This is one value of $x$, from which the others are easily found (Ch. 39). We thus perceive clearly how it is that the value of $y$ may be real, although the sum of two impossible quantities.

(7.) Returning to the determination of $\sqrt[3]{(a\sqrt{g}-b\sqrt{h})}$, —when much difficulty arises in distributing the factors of either term, we may proceed by the following general method, by means of which also we determine at once if the question is capable of a proper solution or otherwise.

If $\sqrt[3]{}(a\sqrt{g} \pm b\sqrt{h}) \doteq \sqrt{x} \pm \sqrt{y}$, ($\sqrt{g}$ and $\sqrt{h}$ being one or both real surds, and also $\sqrt{x}$ and $\sqrt{y}$). Then (Th$^s$ 7 and 11) $\sqrt[3]{}(a\sqrt{g} \mp b\sqrt{h}) \doteq \sqrt{x} \mp \sqrt{y}$, reading the upper sign in the one eq$^n$ with the upper in the other and the lower with the lower. Then, taking the products of like sides of the two eq$^{ns}$,

$$\sqrt[3]{}(a^2g - b^2h) \doteq x - y.$$

Now, since $x$ and $y$ are both rat$^l$, $\sqrt[3]{}(a^2g - b^2h)$ is rat$^l$, and $a^2g - b^2h$ is a complete cube, whenever $\sqrt[3]{}(a\sqrt{g} \pm b\sqrt{h})$ can be expressed as a binomial q$^e$ surd; and if $a^2g - b^2h$ is not a complete cube, we need proceed no further by any method, for a true solution is then impossible. Thus, in Example 4, since $121 - 175, = -54$, is not a complete cube, we might have known, before commencing, that the question could not be resolved.

But when $a^2g - b^2h$ is a complete cube, call it $c^3$. Then $x - y \doteq c$, and $y = x - c$. Cubing both sides of our first eq$^n$, we have

$$a\sqrt{g} \pm b\sqrt{h} = (x - 3y)\sqrt{x} \pm (3x + y)\sqrt{y}.$$

$\therefore$ (Th. 3 or 8) $a\sqrt{g} \doteq (x + 3y)\sqrt{x} = (x + 3x - 3c)\sqrt{x} = (4x - 3c)\sqrt{x}$, or

$$4x^{\frac{3}{2}} - 3cx^{\frac{1}{2}} \doteq a\sqrt{g}.$$

Resolving this as a cubic eq$^n$, or by trials, we find the value of $x^{\frac{1}{2}}$, and thence that of $x$ and $y$.

It is needless to say, however, that this method ought never to be taken if a solution can be found, without much trouble, by the simpler mode previously described.*

# CHAPTER XXXIX.

## THE ROOTS OF EQUATIONS.

**Def.** In any equation involving one unknown quantity, any known quantity which, when substituted for the unknown, satisfies the equation, is called a *Root* of the equation. By *satisfying* the equation we mean producing an identical equation. The roots of an equation are therefore simply the different values of the unknown quantity. Thus 2 and 3 are the roots of the eq$^n$ $x^2-5x+6=0$, for either of these, substituted for $x$, satisfies the equation.

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* We have another method given in some works of high authority, applicable to the cube or any other root; but where it proceeds on the assumption that $2x^3 + 2y^3$ is always a whole number, *that* is not true, and where it is not assumed but taken merely as a particular case, we have no means given of determining beforehand *whether* it is so or not.

## THEOREM.

*Of Equations involving one unknown Quantity, every Simple Equation has one Root and one only, every Quadratic has two (real or imaginary) and only two, and every Cubic Equation has three (real or imaginary) and not more.*

DEM. (1.) Every such simple eq$^n$, when reduced to its simplest form, may be expressed as $x-a=0$, of which $a$ is a root, for $a-a\doteq 0$; and it can have no other root; for, if possible, let that be $b$. Then $b-a\doteq 0$, or $b=a$.

(2.) Every such q$^e$ eq$^n$, reduced to its simplest form, may be expressed as $x^2+ax+b=0$, $a$ and $b$ being either positive or negative. This, resolved by the usual rule, gives $x=-\frac{1}{2}a\pm\frac{1}{2}\sqrt{(a^2-4b)}$. One root, ∴, is $-\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-4b)}$, and another $-\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-4b)}$. Besides these there can be no other; for, if possible, let it be $t$, the two roots we have already found being named $r$ and $s$. Adding together the values of $r$ and $s$ we have $r+s=-a$, or $r+s+a=0$. Then substituting $r$ and $t$ successively for $x$ in the given eq$^n$, we have $r^2+ar+b=0$, and $t^2+at+b=0$. Subtracting the latter of these from the former we find $r^2-t^2+ar-at=0$, or $(r-t)(r+t+a)=0$, an eq$^n$ which is satisfied by making either $r-t=0$, or $r+t+a=0$. If the former, then $r\doteq t$, and if the latter, then $r+s+a\doteq r+t+a$ (since each $\doteq 0$) and $t=s$; so that $t$ must be $=$ either $r$ or $s$, and there cannot be a third root.

(3.) Any cubic eq$^n$ involving only one unknown q$^{ty}$, when reduced to its simplest form, may be expressed thus, $x^3+ax^2+bx+c=0$, $a$, $b$, and $c$ being positive or negative. Of this we have seen how to find one root, in Ch. XXX. Pr$^s$ 2 and 3. Calling that root $r$, and substituting it for $x$ in the given eq$^n$, we have

$$r^3+ar^2+br+c=0.$$

Subtracting this from the given eq$^n$, we have

$$(x^3-r^3)+a(x^2-r^2)+b(x-r)=0, \text{ or}$$
$$(x-r)(x^2+rx+r^2+ax+ar+b)=0.$$

This eq$^n$ will be satisfied by making either factor $=0$. If the first, we have $x=r$; and if the second, we then have the q$^e$ eq$^n$ $x^2+(r+a)x+r^2+ar+b=0$, from which we find two other values of $x$ (say $s$ and $t$), making in all three roots of the given eq$^n$.

(4.) There cannot be more than three; for, if there is a fourth, call it $u$. Then, from the q$^e$ eq$^n$ in the preceding paragraph, since the coef$^t$ of the second term is $a+r$, we have, as proved in (2), $(a+r)+s+t=0$.

Inserting $u$ for $x$ in the given eq$^n$, we obtain

$$u^3+au^2+bu+c=0.$$

Subtracting this from the eq$^n$

$$r^3+ar^2+br+c=0, \text{ we have}$$
$$(u^3-r^3)+a(u^2-r^2)+b(u-r)=0, \text{ or}$$
$$(u-r)(u^2+ur+r^2+au+ar+b)=0.$$

This is satisfied by making either factor $=0$. From the former we obtain $u=r$, and from the latter, $u^2+(r+a)u+r^2+ar+b=0$. But, with $u$ for $x$, this is the same eq$^n$ which we had before, in (3), of which we found the two roots to be $s$ and $t$, *and no other*. There cannot then be a fourth root.

NOTE. Although we speak of two roots to every quadratic, and three to every cubic equation, we have not proved that two of these may not be equal; for, in fact, two of them may be so, or even, in the cubic eq$^n$, all three.*

COR. 1. If $r$ be one root of the q$^e$ eq$^n$ $x^2+ax+b=0$, or of the cubic eq$^n$ $x^3+ax^2+bx+c=0$, we have proved that the left-hand member is, in either case, divisible by $x-r$ without a remainder.

COR. 2. The roots of the q$^e$ eq$^n$ $x^2+ax+b=0$, being $r$ and $s$, $(x^2+ax+b)\div(x-r)\doteq x-s$. For, if we actually divide, we have for our quotient $x+(a+r)$, with $r^2+ar+b$ for a remainder. But we have before proved (2) that $a+r\doteq -s$ and $r^2+ar+b=0$. $\therefore (x^2+ax+b)\div(x-r)\doteq(x-s)$.

COR. 3. Hence $x^2+ax+b\doteq(x-r)(x-s)$.

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* This may appear to be contradictory to the general statement. For if two roots of a quadratic equation are equal it may be alleged that it has not two but one, and similarly of the cubic equation. In like manner it may be said that when the root is imaginary it does not exist, and cannot be counted. In one sense these objections are valid. But it is only *in particular cases* that two or more roots are equal, or any of the roots imaginary, and, by the hypothesis of equal or imaginary roots, we are enabled to express our theorem generally.

COR. 4. Since $(x-r)(x-s) \doteq x^2-(r+s)x+rs = x^2 + ax + b$, we have $-(r+s)x+rs = ax+b$, and it has been proved, in (2), that $-(r+s) \doteq a$; $\therefore rs \doteq b$. Therefore, in any q$^{c}$ eq$^{n}$ of the form $x^2+ax-b=0$, if we take the two roots with their signs changed, the coef$^{t}$ of the second term is their sum, and that of the third term is their product.

COR. 5. Taking $r$, $s$, and $t$ as the roots of the eq$^{n}$ $x^3+ax^2+bx+c=0$, $(x^3+ax^2+bx+c) \div (x-r) \doteq (x-s)(x-t)$. For, if we actually divide, we obtain $x^2+(a+r)x+r^2+ar+b$ for our quotient, with $r^3+ar^2+br+c$, which we have shown (3) to be $=0$, for a remainder. But $x^2+(a+r)x+r^2+ar+b=0$ is the very same eq$^{n}$ from which we previously obtained the roots $s$ and $t$. $\therefore$, by Cor. 4, $x^2+(a+r)x+r^2+ar+b \doteq (r-s)(r-t)$, and $(x^3+ax^2+bx+c) \div (x-r) \doteq (r-s)(r-t)$.

If, $\therefore$, we have found one of the roots, $r$, of any cubic eq$^{n}$ of the form $x^3+ax^2+bx+c=0$, and divide the left-hand member by $x-r$, and if we put the quotient $=0$, we shall have a q$^{c}$ eq$^{n}$ from which we may obtain the other two roots.

COR. 6. Hence $r$, $s$, and $t$, being the roots of the supposed cubic eq$^{n}$,

$$x^3+ax^2+bx+c \doteq (x-r)(x-s)(x-t).$$

COR. 7. On the same understanding, since, as we have shown, $s$ and $t$ are also the roots of the q$^{c}$ eq$^{n}$ $x^2+(a+r)x+r^2+ar+b$, it follows, from Cor. 4, that $s+t \doteq -(a+r)$ and $\therefore r+s+t = -a$.

Also $rs+rt = r(s+t) \doteq -(ar+r^2)$, and (Cor. 4) $st = r^2+ar+b$. $\therefore rs+rt+st \doteq b$.

Again, since $st \doteq r^2+ar+b$, $rst \doteq r^3+ar^2+br$; and we have seen that $r^3+ar^2+br+c \doteq 0$. $\therefore rst \doteq -c$.

Therefore, in any cubic eq$^{n}$ of the form $x^3+ax^2+bx+c$, if we take the three roots with their signs changed, the coef$^{t}$ of the first term is their sum, the coef$^{t}$ of the last term is their product, and the coef$^{t}$ of the second term is the sum of their products taken two and two. The same rule was found to hold in a q$^{c}$ eq$^{n}$, and in fact it is true with regard to equations of all degrees; but into these generally the limits of this volume forbid us to enter.

COR. 8. A quadratic eq$^{n}$ may be formed whose roots shall be any two given quantities as $r$ and $s$, by multiplying $x-r$ by $x-s$, and making the product $=0$; and a cubic eq$^n$ may be formed whose roots shall be any three given quantities, as $r$, $s$, and $t$, by taking the product of $x-r$, $x-s$, and $x-t$, and making it $=0$.

**Example 1.** Form a cubic eq$^n$ whose roots shall be $-2$, $+2$, and $-3$.

$$(x+2)(x-2)(x+3)=x^3+3x^2-4x-12=0.$$

**Example 2.** Having given the eq$^n$ $x^3-9x^2+20x-12=0$, and having found (page 213) one of its roots to be 1, it is required now to determine the others, by Cor. 5.

$$(x^3-9x^2+20x-12)\div(x-1)\doteq x^2-8x+12.$$

Then, from the eq $x^2-8x+12=0$, we find $x=6$ or 2. We may verify these values by Cor. 6, 7, or 8. Thus

$$(x-1)(x-2)(x-6)\doteq x^3-9x^2+20x-12.$$

In applying Cor. 2 or 5, as in the last example, when the sign of the given root is negative, as $-r$, we must divide by $x+r$, which is equivalent to $x-(-r)$; and when, in the case of a q$^e$ eq$^n$, the quotient appears in the form $x+r$, the required root is $-r$.

**Exercise A.** Form a q$^e$ eq$^n$ with its roots 12 and 13; another with its roots $+5$ and $-5$, and a third with its roots $3-\surd 5$ and $3+\surd 5$.

B. Form a cubic eq$^n$ with its roots 4, 5, and 6; a second with its roots $-4$, $-5$, and $-6$, and a third with its roots 10, $1-\surd 2$, and $1+\surd 2$.

1. Having given the eq$^n$ $x^2-\frac{5}{6}x+\frac{1}{6}=0$, and one of its roots $=\frac{1}{2}$, determine the other by Cor. 2. Ans. $\frac{1}{3}$.

2. The same with the eq$^n$ $y^2-y-132=0$, one root being 12. Ans. $-11$.

3. The same with the eq$^n$ $z^2+6z-315=0$, one root being $-21$. Ans. $+15$.

C. Having given the q$^e$ eq$^{ns}$ $x^2-35x+306=0$, $y^2+7y=60$, and $z^2-17z-200=0$, and having found one value of $x$, $y$, and $z$, respectively, $+17$, $+5$, and $-8$, what is the second value of each?

D. One root of the q$^e$ eq$^n$ $x^2-8x+14=0$ is $4+\surd 2$. What is the other?

4. One root of the cubic eq$^n$ $x^3+5x^2-2x-24=0$, is $-3$. What is the product of the other two, and what *their* sum? Ans. $-8$ and $-2$.

E. In the cubic eq$^n$ $y^3 - 16y^2 + 65y - 50$, one of the roots is 1. What are the product and sum of the other two?

F. Two of the roots of the eq$^n$ $z^3 - 5z^2 - 36z + 180 = 0$, are $+6$ and $-6$. What is the third?

5. Of the eq$^n$ $x^3 - 12x^2 + 47x = 60$, one root is 3. What are the two others? Ans. 4 and 5.

6. Of the eq$^n$ $y^3 - y^2 - 14y + 24 = 0$, one root is $-4$. What are the others? Ans. $+2$ and $+3$.

7. One root of the equation $z^3 - 9z^2 + 19z - 5 = 0$, is 5. What are the others? Ans. $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

8. One root of the equation $x^3 - 6x^2 + 5x - 30 = 0$, is 6. Determine the other two.

Ans. $+\sqrt{(-5)}$ and $-\sqrt{(-5)}$.

G. Having given the four cubic eq$^{ns}$, $v^3 - 3\frac{1}{2}v^2 + 3\frac{1}{2}v = 1$, $x^3 - 5x^2 - 72x = 144$, $y^3 + 6y^2 - 51x = 280$, and $z^3 - 20z^2 + 126z = 260$, of which one root of each has been found, viz. $v = +2$, $x = +12$, $y = -8$, and $z = +10$. The other roots are required.

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# CHAPTER XL.

## ALGEBRA APPLIED TO GEOMETRY.

(*Continued from Chapter* XXV.)

In the following exercises the answer is expected in an Algebraical form when surds are involved in it, and free from denominators containing surds. (See Ch. XXXVIII. Pr. 5.)

**Example.** The sum of the base and altitude of an eq$^l$ triangle is 10. What is the base alone?

If we put $2x$ for the length of one side of the eq$^l$ triangle and draw a perp$^r$ from the vertex dividing it into two r$^t$-angled triangles, in each of these we have the hyp$^e$ $= 2x$, the base $= x$, and conseq$^y$ the perp$^r$ $= x\sqrt{3}$.

$$\therefore 2x + x\sqrt{3} = x(2 + \sqrt{3}) \doteq 10\text{; and}$$

$$x \doteq \frac{10}{2+\sqrt{3}} = \frac{10(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})} = \frac{10(2-\sqrt{3})}{4-3}$$

$$= 10(2 - \sqrt{3}).$$

$$\therefore 2x \doteq 20(2 - \sqrt{3}).$$

**Exercise 1.** The side and diagonal of a square are together = 100. What is the length of the side?

Ans. $100(\sqrt{2}-1)$.

A. The radius of a circle circ$^{d}$ about a hexagon exceeds that of a circle insc$^{d}$ in it by 10. What is the side of the hexagon?

2. An octagon is formed by cutting off the corners of a square whose side measures $a$ inches. What is the side of the octagon? Ans. $(\sqrt{2}-1)a$.

B. The diam$^{r}$ of a semicircle is 20. What is the side of a square inscribed in it, one side of the square coinciding with the diam$^{r}$?

3. One square is inscribed in another in such a manner that each corner of the inner square is in the middle of one side of the outer square. The circumf$^{e}$ of the outer square exceeds that of the inner by 100. What is the side of the outer square? Ans. $25(2+\sqrt{2})$.

C. The side of an eq$^{l}$ triangle is 10. Find the side of a square insc$^{d}$ in it in such a way that one side of the square lies on one side of the triangle.

4. Two points are taken in the arc of a semicircle, whose distances from the one extremity of the diameter are 312 and 360, the sum of their distances from the other extremity being 384. What is the diameter? Ans. 390.

D. The two parallel chords of a circular zone are 12 and 16, and its breadth 14. What is the diameter of the circle?

5. Find the side of an eq$^{l}$ triangle insc$^{d}$ in a circle whose radius is $r$, and that of another circ$^{d}$ about the same circle.

Ans. $r\sqrt{3}$ and $2r\sqrt{3}$.

E. Having given the side of an eq$^{l}$ triangle, to find the radii of its insc$^{d}$ and circ$^{d}$ circles.

6. Find the sides of a rectangle, the perimeter of which shall be equal to that of a square whose side is $a$, and its area one fourth of that of the square.

Ans. $a(1+\frac{1}{2}\sqrt{3})$ and $a(1-\frac{1}{2}\sqrt{3})$.

F. Having given the perimeter of a rhombus = 12, and the sum of its two diagonals = 8, to find the diagonals.

7. The side of an octagon being 1, what are the radii of the insc$^{d}$ and circ$^{d}$ circles?

Ans. $\frac{1}{2}(1+\sqrt{2})$ and $\sqrt{(1+\frac{1}{2}\sqrt{2})}$.

G. What is the side of a square insc$^{d}$ in a triangle (as in Ex. C), the altitude of the triangle being $a$ and the base $b$?

8. A st$^{t}$ line $a$ being divided so that the rectangle con-

tained by the whole line and one of the parts is equal to the square of the other part, compute the lengths of the two parts. Ans. $\frac{1}{2}a(\surd 5 - 1)$ and $\frac{1}{2}a(3 - \surd 5)$.

H. The diam$^r$ of a semicircular plot of ground being 30 feet, in what part of the arc must a person stand so as to be twice as far from the one extremity of the diameter as from the other?

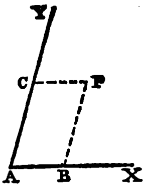
9. What must be the length and breadth of a rectangle whose perimeter and area shall each be equal to those of an eq$^l$ triangle whose side is $a$?

Ans. $\frac{1}{4}a\{3 + \surd(9 - 4\surd 3)\}$ and $\frac{1}{4}a\{3 - \surd(9 - 4\surd 3)\}$.

K. The legs of a r$^t$-angled triangle are $a$ and $b$. What is the radius of the insc$^d$ circle?

In this and Ch. xxv., we have merely, as yet, obtained, by the ordinary processes of Algebra, the lengths of certain lines, from which, when it is required, we may easily construct the figure geometrically. But there remains another most important branch of the subject, which regards the determination not of the lengths but of the positions and directions of lines, and these not merely straight but curved. This is done by obtaining a general algebraical expression for the distance of every point in any such line from each of two given intersecting straight lines.

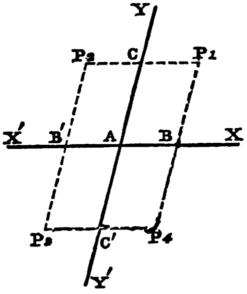
## DEFINITIONS AND EXPLANATIONS.



Let AX and AY be two given diverging lines, and P any point; and let PC and PB be drawn parallel to AX and AY. The position of P is determined if we know the lengths of PC and PB. Suppose, for instance, that these were 2 and 3 inches, we should set off AB and AC of these lengths, and, through C and B, draw st$^t$ lines parallel to AX and AY, intersecting in P. This would fix the place of P. Or, if we know the position of P, the lengths of PC and PB are determined.

The two fixed lines, AX, AY, are called the *Axes* to which the position of P is referred; while the distances PC and PB, or their equals AB and AC, are called the *Co-ordinates* of the point P to the axes AX and AY. The point A, in which the axes meet, is called the *Origin* of the co-ordinates. Sometimes one of the co-ordinates is called the *Abscissa*.

The co-ordinates are usually designated by the letters $x$ and $y$, $x$ being taken as the abscissa when any distinction is made. Thus, in the case previously supposed, $x = 2$ and $y = 3$.



The point P is not necessarily in the angle between AX and AY; for, if we produce these lines to X′ and Y′, P may be in any one of the four angles at A. But to distinguish its place in these angles, we must have recourse to difference of sign in expressing the lengths of the co-ordinates. Thus, if all lengths measured on the axis AX, from A towards X, be expressed by positive numbers, those measured the opposite way, towards X′, will be expressed by negative numbers; and if those measured on AY, towards Y, are regarded as positive, then those measured towards Y′ are regarded as negative. Thus, if $AB' = AB$ and $AC' = AC$, in length; then, if we speak of AB as $+2$, AB′ is $-2$; and if AC is $+3$, AC′ is $-3$. Then the position of the point $P_1$ is indicated by the equations $x = +2$, $y = +3$; that of $P_2$, by $x = -2$, $y = +3$; that of $P_3$, by $x = -2$, $y = -3$; and that of $P_4$, by $x = +2$, $y = -3$.

It is immaterial in which of the two directions we regard the lengths as positive; but, having fixed upon one, we must, throughout any particular investigation, adhere to the choice made, estimating all the lengths expressed positively in one direction, and all expressed negatively in the opposite direction. It is usual, however, to set off the lines represented by positive numbers to the right hand and upward, and those represented by negative numbers to the left and downward.

The axes may either be rectangular (that is, perp$^r$ to each other) or oblique. The former case usually involves the most simple considerations, and to that case we shall confine ourselves, for the most part, in what we have to say on the subject in this volume.

To determine the position of a point with reference to two given axes, by means of its co-ordinates, we must have two equations, as $x = a$ and $y = b$, or $x + y = a$ and $x - y = b$. If one equation only is given containing both $x$ and $y$, *then, for* any given value of $x$ we should have a corresponding value of $y$ (or more than one if the equation is higher than the first degree); but without such limitation we should evidently have an unlimited number of values of each, indicating the positions of an unlimited number of points of intersection of the co-ordinates. If we assign successive values to $x$, we shall obtain successive values to $y$, and successive points of intersection; and if these values be taken indefinitely near to each other, the points of intersection will also be indefinitely close together, and if connected by a continuous line, that line will be either a straight line or what is called a *Regular Curve.*

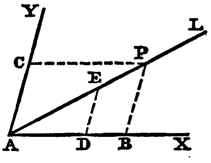
Every single equation involving both $x$ and $y$ will thus be sufficient to enable us to trace out some line, straight or curved, such line being called the *Locus** of the point whose position is indicated by the co-ordinates $x$ and $y$; and every st' line, or regular curve, may have its locus expressed by some equation.

A straight line, as we shall see presently, is always expressed by a simple equation, and a circle by a quadratic equation. In general, lines are said to be *of the first, second, or other order,* according to the degree of the equation which expresses the relation of their co-ordinates.

Since, when the degree of the equation is higher than the first, every value of $x$ must have two or more corresponding values of $y$, and conseq' two or more corresponding points of intersection, every equation above the first degree gives two or more curve lines, or rather what are called *Branches* of the same curve. These branches may either run into each other, forming, when complete, only one curve, or continue separate.

### PROBLEM I.

*To find the Equation for a given Straight Line, referred to any given Axes.*



CASE 1. *When the given line passes through the origin.*

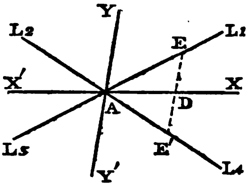
Let AL be the given line, and AX, AY the two given axes. On AX set off AD = 1, and draw DE par' to AY. Call DE, which is given, $n$.

---

* This word is not happily chosen, and probably meant, on its first introduction the trace of the *successive* loci of the same point.

Then, from similar triangles, AD : DE :: AB : BP, or $1 : n :: x : y$. $\therefore y \doteq nx$, which is the equation required.

For every different point in AL, $x$ and $y$ will change their values, while $n$ will continue constant; but $n$ will have different values for different lines, depending upon the angle of the line's inclination to either axis. The same constant $n$ applies both to the line AL, and to its prolongation downward beyond A, as $AL_3$. If $n$ have a positive value, then after setting off $AD = 1$, DE is measured upward from D. In this case when $x$ has a positive value, $nx$ or $y$ is also positive, and the line will extend from A upward to the right, as $AL_1$, but when $x$ becomes negative, $nx$, or $y$, also becomes negative, and the line extends from A to the left downward, as $A_3$. But if $n$ have a negative value it is set off from D downward, as DE′; and, in that case, when $x$ is positive we have $nx$, or $y$, negative, and the line $AL_4$ extending from A to the right downward; but when $x$ also is negative, $nx$, or $y$, is positive, and we have a line, as $AL_2$, extending from A to the left upward.



COR. 1. If the given line coincides with AY, then $x \doteq 0$; and if with AX, then $y \doteq 0$.

EXERCISE A. If $n \doteq +1$ and the axes rectangular, what is the direction of the line?

B. If the axes are rect$^r$ and the given line inclined to AX at angle of 60°, what is $n$?

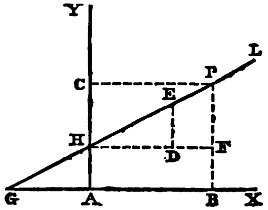
C. If the axes are rect$^r$ and $x = y\sqrt{3}$, how does the line lie?

D. If the axes are rect$^r$ and the given line inclined to AX at an angle of 150°, what value has $n$?

E. If the axes are inclined to each other at an angle of 60°, the given line bisecting that angle, what is $n$?

CASE 2. *When the given line does not pass through the origin*, as HL. Setting off HD and DE as before, we find, in this case, $PF = nx$. But $PB \doteq PF + FB$. That is, putting $a$ for AH,

$$y \doteq nx + a.$$



The direction of the line, it will be observed, determines $n$; its position, $a$; and the place of any given point in the line, determines the values of $x$ and $y$ for that particular point.

Cor. 2. It will be perceived that as the direction and position of the line determine the values of $n$ and $a$, so if the values of $n$ and $a$ are given, we may, from these, draw the line in the direction and position which these values indicate. In Case 2 we may have not only $n$ and $x$ either positive or negative, but also $a$. When $a$ is negative the line HL cuts AY below A.

Cor. 3. If HL is par$^l$ to AX, then $n$ is 0, and $y = a$. If HL is par$^l$ to AY, then $a$ is infinite, $n$ infinite, and $y$ infinite.

Cor. 4. In Case 1, $y \propto x$. In Case 2, $y - a \propto x$.

Exercise F. The axes being rect$^r$, and $a = +10$, where will the line cut the axis AX, 1st, when $n \doteq +\frac{1}{2}$, and 2d, when $n \doteq -\frac{1}{2}$?

G. The same question when $a \doteq -10$.

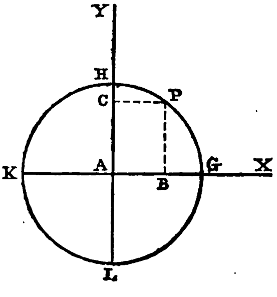
H. According to the data in Ex. F, what will be the value of $y$ and what the position of P, according as $x$ is $+10$, $-10$, $-20$, or $-30$, 1st, when $n$ is $+\frac{1}{2}$, and 2d, when $n$ is $-\frac{1}{2}$?

K. The same when $a$ is $-10$?

L. Find the value of $n$, and show how the line must be drawn, if $3y \doteq x\sqrt{3} \pm 1$.

## Problem II.

*To find the Equation to a given Circle referred to rectangular Axes passing through the Centre.*



Putting $r$ for the radius, and $x$ and $y$, as before, for AB, and AC or BP. From Euc. I. 47, $AP^2 \doteq AB^2 + BP^2$. That is, $r^2 \doteq x^2 + y^2$, or

$$y^2 \doteq r^2 - x^2.$$

Cor. 1. Since $y \doteq \sqrt{(r^2 - x^2)}$, we have for every value of $x$, two values of $y$, one positive, the other negative, and conseq$^y$

two positions of P, one above KG, the other below it; and when B is between A and K, $x$ is itself negative.

EXERCISE A. From the equation just found prove that $PB^2 = KB \times BG$.

B. Prove, from the equation, that the upper branch of the curve and the lower meet in G and K.

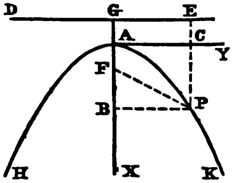
COR. 2. In a similar manner we find for the equation when the origin is at any point in the circumference, $y^2 = 2rv - v^2$, the co-ordinates being $v$ and $y$.

## PROBLEM III.

### *To find the Equation for a given Parabola.*

DEFINITIONS. A *Parabola* is a curve line such that any point in it is at equal distances from a given point (called the *Focus*) and a given straight line (called the *Directrix*).

Let HK be a parabola, F its focus, DE its directrix, and P any point in it. Join PF and draw PE perp^r to DE. Then, by the def^n, PF = PE. Draw FG perp^r to DE, and bisect it in A. AF, produced downwards, is called the *Axis* of the parabola.



Let A be taken as the origin. Put AG or AF $= a$, AB $= x$, and AC or BP $= y$. Then BF $= x - a$, and FP = PE = BG $= x + a$.

But (Euc. I. 47) $BF^2 + BP^2 = FP^2$, or $(x-a)^2 + y^2 = (x+a)^2$. That is, $x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$.

$$\therefore y^2 = 4ax.$$

COR. 1. Hence $y^2 \propto x$, and $y \propto \sqrt{x}$.

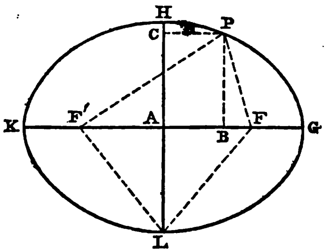
EXERCISE. Prove from the equation that the curve has two branches, one on each side of the axis, and that these meet in A. Also that A is the *Vertex*, or, in other words, that no point in the curve is nearer to the directrix than A.

COR. 2. Since, however large $x$ may be, we obtain two corresponding values of $y$, both possible, the curve has no termination downward. Also since, as $x$ increases, $y$ increases, the curve will continue to deviate further and further from the axis as it is continued downward.

## PROBLEM IV.

*To find the Equation to a given Ellipse referred to rectangular Axes passing through the Centre.*

DEFINITIONS. An *Ellipse* is a curve line such that the sum of the distances of any point in it from two given points (called the *Foci*) is always the same.



Let GHKL be an Ellipse, with the foci F and F′; and let P be any point in it. Then, by the defn, FP+F′P is a constant length, which we shall call $2a$, and which must, of course, be > FF′.

If we bisect FF′ in A, A is called the *Centre*. Put AF or AF′ = $e$, which is called the *Eccentricity*. From P draw PC and PB parl to the axes, and call these (or their equals, AB and AC) $x$ and $y$. Then $BF' = e+x$, and $BF = e-x$.

$PF' = \sqrt{(BF'^2+BP^2)} = \sqrt{\{(e+x)^2+y^2\}}$, and $PF = \sqrt{(BF^2+BP^2)} = \sqrt{\{(e-x)^2+y^2\}}$. But $PF'+PF = 2a$. $\therefore \sqrt{\{(e+x)^2+y^2\}} + \sqrt{\{(e-x)^2+y^2\}} = 2a$; or $\sqrt{\{(e+x)^2+y^2\}} = 2a - \sqrt{\{(e-x)^2+y^2\}}$; and, squaring both sides, $e^2+2ex+x^2+y^2 = 4a^2+e^2-2ex+x^2+y^2-4a\sqrt{(e^2-2ex+x^2+y^2)}$. Cancelling and transposing, $4ex-4a^2 = -4a\sqrt{(e^2-2ex+x^2+y^2)}$; or $a^2-ex = a\sqrt{(e^2-2ex+x^2+y^2)}$. Then, squaring both sides, we have $a^4-2a^2ex+e^2x^2 = a^2e^2-2a^2ex+a^2x^2+a^2y^2$. $\therefore a^4-a^2e^2-a^2x^2+e^2x^2 = a^2y^2$, or $(a^2-e^2)a^2-(a^2-e^2)x^2 = a^2y^2$. If we now put $b^2$ for $a^2-e^2$, we have $b^2(a^2-x^2) = a^2y^2$.

$$\therefore y^2 = \frac{b^2}{a^2}(a^2-x^2).$$

COR. 1. Hence $y^2 \propto a^2-x^2$ and $y \propto \sqrt{(a^2-x^2)}$.

COR. 2. If $e = 0$, F and F′ coincide with $a$, $b^2 = a^2-e^2 = a^2$, and $b = a$; and the equation becomes $y^2 = a^2-x^2$, which is the eqn to a circle.

COR. 3. Since every value of $x$ gives two values of $y$, there must be two branches of the curve, one above FF′, the other below it, and since the negative value of $y$, apart from its sign, is always equal to the positive, the two branches must be symmetrical.

Cor. 4. When $x = \pm a$, $a^2 - x^2 = 0$, and $\therefore y = \pm 0$. If $\therefore$ AG and AK be set off on FF′ produced, each $= a$, the two branches of the curve will meet in G and K, and the curve is a continuous circumference.

Cor. 5. Since we have the same value of $y$ for $+x$ and $-x$, the right and left sides are symmetrical.

Cor. 6. If the curve cut HL in H and L, AH and AL will be each $= b$. For if we join LF and LF′, these are equal, and being together $= 2a$ (by the defn) each of them is $= a$, and $\therefore \text{AL}^2 = a^2 - e^2 = b^2$, and $\text{AL} = b$; and for a like reason, $\text{AH} = b$.

Def. KG and LH are called the *Transverse and Conjugate Axes of the Ellipse;* or the *Major and Minor Axes,* since $a$, being $= \sqrt{(b^2 + e^2)}$, must always be greater than $b$.

Exercise A. Prove, from the equation, that no part of the curve can be further from the minor axis than G or K, and no part further from the major axis than H or L.

B. Prove, from the eqns to the ellipse and circle, that, if a circle be described on the transverse axis as a diameter, all the ordinates of the ellipse, parl to AH or AL, will have the same ratio to the corresponding ordinates of the circle, that is to those which have the same co-ordinate.

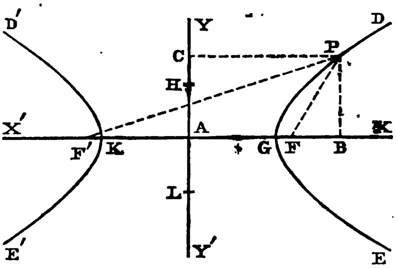
Cor. 7. When the extremity of the transverse axis is made the origin, our equation is $y^2 = \frac{b^2}{a^2}(2av - v^2)$, the co-ordinates being $v$ and $y$. This will be proved by putting $a - v$ for $x$ in the equation found above.

## Problem V.

*To find the Equation to a given Hyperbola referred to rectangular Axes passing through the Centre.*

Definitions. A *Hyperbola* is a curve line such that the difference of the distances of any point in it from two given points (called the *Foci*) is always the same,—that is, not changing as the position of the point in the curve changes.

Let DGE be a hyperbola, with the foci F and F'; and let P be any point in it. Then, by the def$^n$, F'P − FP is a constant length, independent of the position of P, which length we shall call $2a$, and which must, of course, be < FF'.



If we bisect FF' in A, A is called the *Centre*. Put AF or AF' $= e$, which is called the *Eccentricity*. From P draw PC and PB par$^l$ to the axes AX and AY, and call these (or their equals, AB and AC) $x$ and $y$. Then BF' $= x + e$ and BF $= x - e$.

PF' $= \sqrt{(\text{BF}'^2 + \text{BP}^2)} = \sqrt{\{(x+e)^2 + y^2\}}$ and PF $= \sqrt{(\text{BF}^2 + \text{BP}^2)} = \sqrt{\{(x-e)^2 + y^2\}}$. But PF' + PF $= 2a$. $\therefore \sqrt{\{(x+e)^2+y^2\}} - \sqrt{\{(x-e)^2+y^2\}} = 2a$. Proceeding now in the same manner as in the case of the Ellipse, we find $e^2x^2 - a^2x^2 - a^2e^2 + a^4 = a^2y^2$, or $(e^2 - a^2)x^2 - (e^2 - a^2)a^2 = a^2y^2$. If we now put $b^2$ for $e^2 - a^2$, we have $b^2x^2 - b^2a^2 = a^2y^2$, or

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2).$$

Cor. 1. Hence $y^2 \propto x^2 - a^2$ and $y \propto \sqrt{(x^2 - a^2)}$.

Cor. 2. For the same reason as in the case of the ellipse, there must be two branches of the curve, one above AF and the other below it, and the two must be symmetrical.

Cor. 3. We have hitherto supposed F'P to be > FP; but if we take FP as the greater, with the same difference as before, we shall evidently have another curve, D'E', or rather another double branch of the same curve, in every way symmetrical with DE, and called the Opposite Hyperbola.

Cor. 4. When $x = \pm a$, $x^2 - a^2 = 0$, and $\therefore y = \pm 0$. If $\therefore$ AG and AK be set off on FF' produced, each $= a$, the upper branch on each side will meet the lower, and the two double branches of the curve will meet the line FF' in G and K.

Cor. 5. Since $x$ may be as large as we please without rendering the corresponding value of $y$ impossible, all the four branches of the curve are unlimited towards D, E, D′ and E′. And since $y$ increases with every increase of $x$, each double branch of the curve constantly opens wider as it recedes from A.

Cor. 6. If through A we draw YY′ perp^r to XX′ and set off upon it AH and AL each $= b$; then, on joining GH, GL, KH, KL, each of these four lines is $= e$.

Def. KG is called the *Transverse Axis* of the two opposite hyperbolas; and HL their *Conjugate Axis*.

Cor. 7. When the nearer extremity of the major axis is made the origin, our equation is $y^2 = \frac{b^2}{a^2}(v^2 + 2av)$; and when the further extremity, $y^2 = \frac{b^2}{a^2}(z^2 - 2az)$, $v$ and $y$, or $z$ and $y$, being the co-ordinates, that is $v = \text{GB}$, and $z = \text{KB}$. This will be proved by putting $a + v$, or $z - a$, for $x$, in the equation previously found.

Exercise. Prove that the conjugate axis of a hyperbola is not always the minor axis.

END OF PART I.

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